ON THE DEGREE OF APPROXIMATION OF A FUNCTION BY THE PARTIAL SUMS OF ITS FOURIER SERIES(¹)

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ABSTRACT. When f is a 2π periodic function with rth order fractional derivative, $r > 0$, of p-bounded variation, Golubov has obtained estimates of the degree of approximation of f, in the L^q norm, $q > p$, by the partial sums of its Fourier series. Here we consider the analogous problem for functions whose fractional derivatives are of Φ-bounded variation and obtain estimates of the degree of approximation in an Orlicz space norm. In a similar manner we shall extend various results that he obtained on degree of approximation in the sup norm.

§I

Let f be a real function of period 2π. If the rth fractional derivative, $r > 0$, is of bounded p-variation, Golubov [3] has obtained estimates of the degree of approximation of f, in the sup and L^q norms, $q > p$, by the partial sums of the Fourier series of f.

We consider the analogous problem for functions whose fractional derivatives are of Φ-bounded variation. In 2.1 we extend Salem's theorem of uniform convergence of Fourier series. In 2.2 we obtain estimates on the degree of approximation in an Orlicz space norm.

In this chapter we shall review the background material which we shall require. In §1 we concern ourselves with Orlicz spaces, §2 will present basic information on Φ-bounded variation, and §3 contains the relevant information on the fractional integral and derivative.

1. **Orlicz spaces.** In this section we shall summarize some results on Orlicz spaces which we shall require. Unless otherwise noted, a complete presentation of this topic can be found in the texts of Krasnosel'skiĭ and Rutickiĭ [4] and Zygmund [14].

Let φ be a nondecreasing right continuous function, $\varphi\colon R^+ \to R^+$, $\varphi(0) =$

$0, \lim_{x \to \infty} \varphi(x) = \infty$. Define

$$\varphi^{-1}(y) = \sup_{\varphi(x) < y} x.$$

Then φ^{-1} is right continuous, nondecreasing, $\varphi^{-1}: R^+ \to R^+$, $\varphi^{-1}(0) = 0$, $\lim_{y \to \infty} \varphi^{-1}(y) = \infty$. If φ is strictly increasing then the usual definition of φ^{-1} and the above definition coincide.

Let

$$\Phi(x) = \int_0^{|x|} \varphi(t)\, dt \quad \text{and} \quad \Phi^*(x) = \int_0^{|x|} \varphi^{-1}(t)\, dt.$$

Φ and Φ^* are both convex, strictly increasing functions such that

$$\lim_{x \to 0} \Phi(x)/x = \lim_{y \to 0} \Phi^*(y)/y = 0, \quad \lim_{x \to \infty} \Phi(x)/x = \lim_{y \to \infty} \Phi^*(y)/y = \infty,$$

and (Young's inequality): for a, b real numbers,

$$ab \leqslant \Phi(a) + \Phi^*(b),$$

equality occurring only when $b = \varphi(|a|)\text{sgn } a$.

It is known that if M is any convex strictly increasing function such that

$$\lim_{u \to 0} M(u)/u = 0, \quad \lim_{u \to \infty} M(u)/u = \infty,$$

then there exists a right continuous nondecreasing $m(x)$; $m: R^+ \to R^+$, $m(0) = 0$, $\lim_{x \to \infty} m(x) = \infty$ and

$$M(x) = \int_0^{|x|} m(t)\, dt.$$

All functions which admit this representation are called Young's functions or N-functions.

Henceforth, we consider integrable functions f of period 2π; integration will be with respect to Lebesgue measure and will extend over a full period unless otherwise noted. Define

$$L_\Phi = \left\{ f: \int \Phi(f) < \infty \right\}$$

and

$$L_\Phi^* = \{ f: kf \in L_\Phi \text{ for some real number } k \neq 0 \}.$$

L_Φ is called an Orlicz class. With norm

$$\|f\|_\Phi = \sup\left\{ \int fg: \int \Phi^*(g) \leqslant 1 \right\}$$

L_Φ^* is a Banach space, and is called an Orlicz space.

Luxemburg introduced an alternative norm on L_Φ^*:

$$\|f\|_\Phi' = \inf\left\{ k: \int \Phi(f/k) \leqslant 1, k > 0 \right\}.$$

We have the relation $\| \cdot \|'_\Phi \leqslant \| \cdot \|_\Phi \leqslant 2\| \cdot \|'_\Phi$, making them equivalent norms, but, for $f \neq 0$ a.e., $\|f\|_\Phi / \|f\|'_\Phi = C$, C not dependent on f, if and only if $\Phi(u) = A|u|^p, p > 1$.

It is known that for all $f \in L^*_\Phi$, $g \in L^*_{\Phi^*}$,

$$\int \Phi(f) \leqslant \|f\|'_\Phi \qquad \text{if and only if } \|f\|'_\Phi \leqslant 1,$$

$$\|f\|'_\Phi \leqslant \int \Phi(f) \quad \text{otherwise,}$$

and

$$\int fg \leqslant \begin{cases} \|f\|_\Phi \|g\|_{\Phi^*}, \\ \|f\|'_\Phi \|g\|_{\Phi^*}, \\ \|f\|_\Phi \|g\|'_{\Phi^*}. \end{cases}$$

We introduce the Φ-moduli of continuity of f:

$$\omega_\Phi(\delta; f) = \sup_{|h| < \delta} \|f(x + h) - f(x)\|_\Phi$$

and

$$\omega'_\Phi(\delta; f) = \sup_{|h| < \delta} \|f(x + h) - f(x)\|'_\Phi.$$

Then $\omega_\Phi(0; f) = \omega'_\Phi(0; f) = 0$; ω_Φ and ω'_Φ are nondecreasing. These functions are continuous if f can be approximated in norm by trigonometric polynomials. The definitions given above are analogous to those of $\omega(p)$, p-modulus of continuity of $f \in L^p$ and of ω, the modulus of continuity of $f \in C$.

We say that an N-function Φ satisfies the Δ_2-condition ("is Δ_2") if there exists $k > 0$ such that for $u > 0$, $\Phi(2u) \leqslant k\Phi(u)$.

We say that Φ satisfies the Δ_2-condition for large (small) values if there exists $k, u_0 > 0$ such that for $u \geqslant u_0$ ($u \leqslant u_0$), $\Phi(2u) \leqslant k\Phi(u)$.

The following conditions are equivalent:

1. Φ is Δ_2 (for large values, for small values).

2. For each $c > 1$, there exists $k(c)$ such that $\Phi(cu) \leqslant k(c)\Phi(u)$ for all $u > 0$ ($u \geqslant$ some u_0, $u \leqslant$ some u_0).

3. There exists $\alpha > 1$ such that, if φ is the right continuous right derivative of Φ,

$$|u|\varphi(|u|)/\Phi(u) < \alpha$$

for all $u \neq 0$ (for $|u| \geqslant$ some u_0, for $0 < |u| \leqslant$ some u_0).

From condition 3 we have that if Φ is Δ_2 for large values, $|u| \geqslant u_0$, then

$$\Phi(u) \leqslant \Phi(u_0)|u|^\alpha/u_0^\alpha;$$

if Φ is Δ_2 for small values, $|u| \leqslant u_0$, then

$$\Phi(u) \geqslant \Phi(u_0)|u|^{\alpha}/u_0^{\alpha};$$

and if Φ is Δ_2, both these inequalities hold.

We have, further, that $L_{\Phi}^* = L_{\Phi}$ if and only if Φ is Δ_2 for large values, and L_{Φ}^* is separable if and only if Φ is Δ_2 for large values. When L_{Φ}^* is separable, the polynomials are dense in L_{Φ}^*. Thus, we see that we can guarantee continuity of ω_{Φ} if Φ is Δ_2 for large values; otherwise, continuity depends on the function $f \in L_{\Phi}^*$.

If we have a sequence $\{f_n\}_{n=1}^{\infty} \in L_{\Phi}^*$ then norm convergence to, say, f_0 implies mean convergence to $f_0 \in L_{\Phi}^*$. The converse is true if and only if Φ is Δ_2 for large values.

We define a partial ordering on the set of N-functions by $\Phi_1 < \Phi_2$ if there exists u_0, $a > 0$ such that for $u \geqslant u_0$,

$$\Phi_2(u) \leqslant a\Phi_1(u).$$

$\Phi_1 \sim \Phi_2$ (read "is equivalent to") if $\Phi_1 < \Phi_2 < \Phi_1$. It is known that $\Phi_1 < \Phi_2$ implies that $L_{\Phi_1} \subseteq L_{\Phi_2}$, and, hence, that equivalent N-functions generate the same Orlicz class. Clearly, then, if $\Phi_1 \sim \Phi_2$ and Φ_1 is Δ_2 for large values, then Φ_2 is also. It is known that in each equivalence class of functions which are Δ_2 for large values there is at least one function which is Δ_2.

From the remarks following condition 3 we have that "Φ is Δ_2 for large values" implies that

$$L^{\alpha} = \left\{ f: \int |f|^{\alpha} < \infty \right\} \subseteq L_{\Phi} = L_{\Phi}^*.$$

Ryan [10] has shown that reflexivity of L_{Φ}^* is equivalent to both Φ and Φ^* satisfying the Δ_2-condition for large values.

Orlicz first introduced $(L_{\Phi}^*, \| \cdot \|_{\Phi})$ with the restriction that Φ be Δ_2 for large values; Zygmund removed that restriction.

We say that Φ satisfies the Δ'-condition if there exists $c > 0$ such that for x, $y > 0$,

$$\Phi(xy) \leqslant c\Phi(x)\Phi(y).$$

We say that Φ satisfies the Δ'-condition for large (small) values if there exists $c > 0$ and $u_0 > 0$ such that for $|x|, |y| \geqslant u_0$ ($|x|, |y| \leqslant u_0$)

$$\Phi(xy) \leqslant c\Phi(x)\Phi(y).$$

All functions which are Δ' in one of the above senses are also Δ_2 in that sense, but the converse implication is not true.

If Φ is Δ' (Δ_2) for small values, say $|u| \leqslant u_0$, then we can replace u_0 by any $w > u_0$, but the Δ' (Δ_2) constant increases unboundedly with w unless Φ is Δ' (Δ_2) for all values.

2. Φ-variation.

We shall require only a few basic notions concerning generalized variation. A more detailed presentation is found in the papers of L. C. Young [13], E. R. Love [5], and Musielak and Orlicz [6].

We define, for an N-function Φ,

$$V_\Phi(f; I) = \sup_Q \sum \Phi(f(x_i) - f(x_{i-1})),$$

where the supremum is taken over all partitions Q of the interval I. We say that $f \in \mathcal{V}_\Phi(I)$ if $V_\Phi(f; I) < \infty$.

Define

$$V_\Phi^{(\delta)}(f; I) = \sup_{|Q| < \delta} \sum \Phi(f(x_i) - f(x_{i-1})),$$

where $|Q|$ denotes the mesh of the partition Q, and let

$$V_\Phi^*(f; I) = \lim_{\delta \to 0^+} V_\Phi^{(\delta)}(f; I).$$

It is easily shown that $f \in \mathcal{V}_\Phi(I)$ if and only if $V_\Phi^*(f; I) < \infty$.

We define $f \in \mathcal{V}_\Phi^*(I)$ if there exists $k \neq 0$ such that $(kf) \in \mathcal{V}_\Phi(I)$. The class $\mathcal{V}_\Phi^*(I)$ is linear, but $\mathcal{V}_\Phi(I)$ is linear (and $= \mathcal{V}_\Phi^*(I)$) if and only if Φ satisfies the Δ_2-condition for small values.

If the interval with which we are concerned is $[0, 2\pi]$, or any period, we shall omit all reference to it and write $V_\Phi(f)$, $\mathcal{V}_\Phi^*$, etc.

We define two variational seminorms by

$$\|f\|_{\Phi V} = \sup \left\{ \sum (f(x_i) - f(x_{i-1})) b_i : \sum \Phi^*(b_i) \leq 1, \text{ all } Q \right\},$$

where the supremum is taken simultaneously over all partitions Q and all sequences $\{b_i\}$ such that $\sum \Phi^*(b_i) \leq 1$, and

$$\|f\|_{\Phi V}' = \inf\{k : V_\Phi(f/k) \leq 1\}.$$

Their equivalence is shown by the relation

$$\|\cdot\|_{\Phi V}' \leq \|\cdot\|_{\Phi V} \leq 2\|\cdot\|_{\Phi V}'.$$

Another partial ordering is defined on the set of all N-functions by $\Phi_1(u) \ll \Phi_2(u)$ if there exist $a, u_0 > 0$ such that for $|u| \leq u_0$,

$$\Phi_2(u) \leq a\Phi_1(u).$$

It is known that $\Phi_1 \ll \Phi_2$ implies $\mathcal{V}_{\Phi_1} \subseteq \mathcal{V}_{\Phi_2}$, and thus defines another equivalence relation on the set of N-functions.

Functions f of Φ bounded variation can have simple discontinuities only. We shall assume that they are normalized so that

$$f(x) = \tfrac{1}{2}(f(x +) + f(x -)).$$

3. Fractional integral and derivative. In addition to our previous assumptions concerning the integrability of f, we shall assume that $\int f = 0$.

We may choose the constant of integration so that the mean value of $f_{(1)}$, the integral of f, is zero. Since $f_{(1)}$ is periodic and of mean value zero, when integrating it we can choose the constant of integration so that $f_{(2)}$ is periodic and has mean value zero. In general, having defined $f_{(1)}, \ldots, f_{(r-1)}$, choose for $f_{(r)}$ that primitive of $f_{(r-1)}$ whose mean value is zero, i.e. if

$$f(x) \sim S[f] = {\sum}' c_n e^{inx} = \sum_{n \neq 0} c_n e^{inx},$$

where $S[\cdot]$ represents the Fourier series, then

(∗) $$f_{(r)}(x) \sim {\sum}' c_n e^{inx} (in)^{-r} = \frac{1}{2\pi} \int f(t) D^{(r)}(x - t)\, dt$$

where

(∗)′ $$D^{(r)}(t) = {\sum}' (in)^{-r} e^{int}.$$

For $0 < t < 2\pi$, the function $D^{(r)}(t)$ is a polynomial of degree r. We may consider (∗) to be a definition of $f_{(r)}$, the fractional integral of order r, for all $r > 0$, provided we set

$$(in)^{-r} = |n|^r \exp\left(-\tfrac{1}{2} i\pi r \operatorname{sgn} n\right), \qquad n \neq 0.$$

The series (∗)′ can be written

$$2 \cos \tfrac{1}{2} \pi r \sum_{n=1}^{\infty} n^{-r} \cos nt + 2 \sin \tfrac{1}{2} \pi r \sum_{n=1}^{\infty} n^{-r} \sin nt.$$

We know that this converges for $t \neq 0$ to a sum $D^{(r)}(t)$ and that it is $S[D^{(r)}]$. Thus, the integral in (∗) exists almost everywhere and its value, $f_{(r)}(x)$, is integrable. It is known that the series in (∗) converges almost everywhere and is $S[f_{(r)}]$, and $f_{(r)}$ coincides with the ordinary integral for $r = 1, 2, \ldots$. Thus

$$D^{(r)}(t) = \sum_{n=1}^{\infty} n^{-r} \cos\left(nt - \tfrac{1}{2} \pi r\right),$$

and

$$f_{(r)}(x) = \cos \tfrac{1}{2} \pi r \sum_{n=1}^{\infty} n^{-r} (a_n \cos nx + b_n \sin nx)$$

$$+ \sin \tfrac{1}{2} \pi r \sum_{n=1}^{\infty} n^{-r} (a_n \sin nx - b_n \cos nx).$$

The fractional integral operator is "additive", i.e. $f_{(\alpha+\beta)} = (f_{(\alpha)})_{(\beta)}$, almost everywhere.

Let $r > 0$, and m an integer such that $m - 1 < r < m$. The derivative of f of fractional order r is generally defined by

$$f^{(r)}(x) = d^m f_{(m-r)}(x)/dx^m.$$

If we suppose $0 < r < 1$, and $f_{(1-r)}(x)$ is absolutely continuous, then

$$S[f^{(r)}] = S'[f_{(1-r)}] = \sum (in)^r c_n e^{inx},$$

and, almost everywhere,

$$f(x) = \int D^{(r)}(x - t) f^{(r)}(t) \, dt.$$

In other words, f is the fractional integral of order r of $f^{(r)}$. We shall adopt a more restrictive definition so that the fractional derivative always has this property.

We say that $g(x) \in L^1(-\pi, \pi)$ is an rth order fractional derivative of $f \in L^{-1}(-\pi, \pi)$, $r > 0$, if

(1) $\int f = \int g = 0$,

(2) $f(x) = \int D^{(r)}(x - t) g(t) \, dt$, a.e.

We denote g by $f^{(r)}$; $f^{(0)}$ will denote the function f. This definition is sufficient for our purposes and will be used throughout.

If we assume that f has an rth derivative, g, according to this last definition, then if

$$g(x) \sim \sum' \gamma_n e^{inx},$$

we have, almost everywhere,

$$f(x) = g_{(r)}(x) = \sum' \gamma_n (in)^{-r} e^{inx} = \sum' c_n e^{inx},$$

$$g(x) \sim \sum' (in)^r c_n e^{inx}$$

and $f_{(m-r)} = g_{(m)}$.

If $f(h; x)$ denotes the Steklov function,

$$\frac{1}{h} \int_{-h/2}^{h/2} f(x + t) \, dt,$$

and $0 < m - 1 < r < m$, we have, almost everywhere,

$$(f(h; x))^{(r)} = \frac{1}{h} \frac{d^m}{dx^m} \int_{-h/2}^{h/2} \int D^{(m-r)}(x - t) f(y + t) \, dt \, dy$$

$$= \frac{1}{h} \frac{d^m}{dx^m} \int_{-h/2}^{h/2} \int D^{(m-r)}((x + y) - t) f(t) \, dt \, dy$$

$$= \frac{1}{h} \frac{d^m}{dx^m} \int_{-h/2}^{h/2} f_{(m-r)}(x + y) \, dy$$

$$= \frac{1}{h} \int_{-h/2}^{h/2} f^{(r)}(x + y) \, dy = f^{(r)}(h; x).$$

The functions we shall consider are in class L_Φ^* and have rth fractional derivative in class $\mathcal{V}_\Phi$, $r \geq 0$. Thus, the functions and their rth derivatives have mean value zero. The class $\mathcal{V}_\Phi^{(r)}$, $r \geq 0$, will denote the class of functions f such that $f^{(r)}$ exists, $f^{(r)} \in \mathcal{V}_\Phi$, and,

$$f^{(r)}(x) = \tfrac{1}{2}\left(f^{(r)}(x+) + f^{(r)}(x-)\right).$$

§II

1. Approximation in the C-norm. We devote this section to generalizations of Salem's theorem on uniform convergence [11].

1.1 THEOREM. *If $f^{(r)} \in \mathcal{V}_\Phi^{(r)}$, $0 \leq r < \infty$, $\sum \Phi^*(1/k) < \infty$, $f^{(r)}$ continuous, then $\|f - S_n(f)\|_C = o(n^{-r})$.*

PROOF. For $r = 0$ the conclusion is Salem's theorem so we assume $r > 0$. Define

$$D_n^{(r)} = \sum_{k=n+1}^{\infty} k^{-r}\cos\left(kt - \tfrac{1}{2}\pi r\right) = D_0^{(r)}(t) - S_n\left(D_0^{(r)}; t\right)$$

and

$$\varphi_x(t) = f^{(r)}(x+t) - f^{(r)}(x).$$

Then

$$f(x) - S_n(f; x) = \frac{1}{\pi}\int_{-\pi}^{\pi} D_n^{(r)}(t)\left(f^{(r)}(x-t) - f^{(r)}(x)\right) dt$$

$$= \frac{1}{\pi}\left(\int_{0<|t|<\pi/n} + \int_{\pi/n<|t|<\pi}\right) D_n^{(r)}(t)\varphi_x(-t)\, dt.$$

It is known [9] that

(1) $$\int_{0<|t|<\pi/n} \left|D_n^{(r)}(t)\right| dt = O(n^{-r});$$

hence,

(2) $$\frac{1}{\pi}\int_{0<|t|<\pi/n} D_n^{(r)}(t)\varphi_x(-t)\, dt = o(n^{-r}),$$

since $f^{(r)}$ is continuous.

We follow [8] and perform an Abel transformation on the remaining integral, which yields

$$\frac{1}{\pi}\int_{\pi/n<|t|<\pi} D_n^{(r)}(t)\varphi_x(-t)\, dt$$

(3) $$= \frac{(-1)}{\pi(n+1)^r}\int_{\pi/n<|t|<\pi} A_n^{(r)}(t)\varphi_x(-t)\, dt$$

$$+ \frac{1}{\pi}\sum_{k=n+1}^{\infty}\left(k^{-r} - (k+1)^{-r}\right)\int_{\pi/n<|t|<\pi} A_k^{(r)}(t)\varphi_x(-t)\, dt$$

where

$$A_k^{(r)}(t) = \sin\left(\frac{2k+1}{2}t - \frac{1}{2}r\pi\right)\Big/2\sin\frac{t}{2}$$

(4)
$$= \cos\frac{1}{2}r\pi\sin\left(\frac{2k+1}{2}t\right)\Big/2\sin\frac{t}{2}$$

$$+ \sin\frac{1}{2}r\pi\cos\left(\frac{2k+1}{2}t\right)\Big/2\sin\frac{t}{2}.$$

We want to show that

$$(5) \qquad \int_{\pi/n<|t|<\pi} A_k^{(r)}(t)\varphi_x(-t)\,dt = o(1)$$

uniformly in x as $n \to \infty$ $(k \geqslant n)$. Now let

$$\int_{\pi/n<|t|<\pi} A_k^{(r)}(t)\varphi_x(-t)\,dt = \cos\frac{1}{2}r\pi\, J_n^{(k)}(t) + \sin\frac{1}{2}r\pi\, H_n^{(k)}(t).$$

Letting D_k denote the kth Dirichlet kernel and observing that $k \geqslant n$,

$$J_n^{(k)}(x) = \int_{\pi/n<|t|<\pi} D_k(t)\varphi_k(-t)\,dt$$

(6)
$$= \int_{\pi/n<|t|<\pi} \frac{\sin kt}{t}\varphi_x(t)\,dt + o(1)$$

where $o(1)$ is uniform in x as $n \to \infty$.

Let

$$J_{n,+}^{(k)}(x) = \int_{\pi/n<|t|<\pi} \frac{\sin kt}{t}\varphi_x(t)\,dt = L_n^{(k)}(x) + o(1),$$

where $o(1)$ is uniform in x as $n \to \infty$, and

$$L_n^{(k)}(x) = \sum_{j=[k/n]}^{k-1} \int_{j\pi/k}^{(j+1)\pi/k} \frac{\sin kt}{t}\varphi_x(t)\,dt.$$

Letting $kt = u$, we obtain, for some $\theta \in (\pi, 2\pi)$,

$$L_n^{(k)}(x) = \int_{\pi}^{2\pi} \sum_{j=[k/n]-1}^{k-2} (-1)^j\varphi_x\left(\frac{u+j\pi}{k}\right)\frac{\sin u}{u+j\pi}\,du$$

(7)
$$= \pi\sin\theta \sum_{j=[k/n]-1}^{k-2} (-1)^j\varphi_x\left(\frac{\theta+j\pi}{k}\right)\frac{1}{\theta+j\pi}.$$

Without loss of generality, we assume the number of terms in the summation to be even. If the number of terms is odd, we may omit either the first or the last term, both of which are $o(1)$ as $n \to \infty$.

Neglecting signs, the sum of two consecutive terms is

$$\frac{\varphi_x\left(\frac{\theta + h\pi}{k}\right)}{\theta + 2h\pi} - \frac{\varphi_x\left(\frac{\theta + (2h + 1)\pi}{k}\right)}{\theta + (2h + 1)\pi} = \frac{\varphi_x\left(\frac{\theta + 2h\pi}{k}\right) - \varphi_x\left(\frac{\theta + (2h + 1)\pi}{k}\right)}{\theta + 2h\pi}$$

$$+ \varphi_x\left(\frac{\theta + (2h + 1)\pi}{k}\right)\left(\frac{1}{\theta + 2h\pi} - \frac{1}{\theta + (2h + 1)\pi}\right).$$

Now,

$$(8) \qquad \left|\varphi_x\left(\frac{\theta + (2h + 1)\pi}{k}\right)\left(\frac{1}{\theta + 2h\pi} - \frac{1}{\theta + (2h + 1)\pi}\right)\right| < \frac{\omega\left(\frac{(2h + 3)\pi}{k}; f^{(r)}\right)}{(2h + 1)(2h + 2)\pi}.$$

Choosing an integer M, $M = M(n) \simeq \sqrt{n}$, the sum of the first M of these terms is

$$< c\omega(cM/k + o(1); f^{(r)}) = c\omega(o(1); f^{(r)}) = o(1)$$

as $n \to \infty$. The sum of the remaining terms is

$$< \frac{c \sup|f^{(r)}|}{M + [k/n]} = o(1) \quad \text{as } n \to \infty.$$

Considering

$$(9) \qquad \frac{\varphi_x\left(\frac{\theta + 2h\pi}{k}\right) - \varphi_x\left(\frac{\theta + (2h + 1)\pi}{k}\right)}{\theta + 2h\pi}$$

and replacing $\theta + 2h\pi$ by $(2h + 2)\pi$ in the denominator, there is an error

$$(10)$$

$$\left|\varphi_x\left(\frac{\theta + 2h\pi}{k}\right) - \varphi_x\left(\frac{\theta + (2h + 1)\pi}{k}\right)\right|\left|\frac{1}{\theta + 2h\pi} - \frac{1}{(2h + 2)\pi}\right|$$

$$= \left|\varphi_x\left(\frac{\theta + 2h\pi}{k}\right) - \varphi_x\left(\frac{\theta + (2h + 1)\pi}{k}\right)\right|\left|\frac{2\pi - \theta}{(\theta + 2h\pi)(2h + 2)\pi}\right|$$

$$< \left|f^{(r)}\left(x + \frac{\theta + 2h\pi}{k}\right) - f^{(r)}\left(x + \frac{\theta + (2h + 1)\pi}{k}\right)\right|\frac{1}{\pi(2h + 2)(2h + 1)}$$

$$< \frac{\omega(\pi/k; f^{(r)})}{(2h + 2)(2h + 1)} < \frac{\omega(\pi/n; f^{(r)})}{(2h + 2)(2h + 1)\pi}.$$

Thus, the replacement gives an error in the sum of these terms which is less than $c\omega(\pi/n; f^{(r)})$.

For simplicity, we suppose $[k/n] - 1$ is even. It suffices, then, to consider

(11)
$$\sum_{h=([k/n]-1)/2}^{[(k-2)/2]} \frac{f^{(r)}\left(x + \frac{\theta + 2h\pi}{k}\right) - f^{(r)}\left(x + \frac{\theta + (2h+1)\pi}{k}\right)}{(2h+2)\pi}, \quad \pi < \theta < 2\pi.$$

Let

$$\Delta_k^h = \left| f^{(r)}\left(x + \frac{2h\pi + \theta}{k}\right) - f^{(r)}\left(x + \frac{\theta + (2h+1)\pi}{k}\right) \right|$$

and let ε_j be a positive sequence decreasing to zero as slowly as necessary to have $\sum_{k=1}^{\infty} \Phi^*(1/k\varepsilon_k) < \infty$.

For proper choice of $b > 0$,

$$\frac{1}{2\pi} \sum_h \frac{b\Delta_k^h}{(h+1)\varepsilon_{h+1}} < \sum_h \Phi(b\Delta_k^h) + \sum_h \Phi^*\left(\frac{1}{(h+1)\varepsilon_{h+1}}\right) \leqslant A < \infty.$$

Choose $m = m(k, n) < (k-2)/2$. Then

(12)
$$\frac{1}{2\pi} \sum_{h=m}^{[(k-2)/2]} \frac{b\Delta_k^h}{h+1} < A\varepsilon_{m+1}$$

and

(13)
$$\frac{1}{2\pi} \sum_{h=1}^{[(k-2)/2]} \frac{\Delta_k^h}{h+1} < \omega(\pi/k; f^{(r)}) \sum_{h=1}^{m-1} \frac{1}{h+1} + \frac{A\varepsilon_{m+1}}{b}$$

$$< \omega(\pi/k; f^{(r)})\log m + \frac{A\varepsilon_{m+1}}{b}.$$

We choose m increasing with n so that

$$\omega\left(\pi/n; f^{(r)}\right)\log m \to 0.$$

Thus

$$\sum \frac{\varphi_x\left(\frac{\theta + 2h\pi}{k}\right) - \varphi_x\left(\frac{\theta + (2h+1)\pi}{k}\right)}{\theta + 2h\pi} = o(1).$$

Similarly we may show that

$$J_{n,-}^{(k)}(x) = \int_{-\pi}^{-\pi/n} \frac{\sin kt}{t} \varphi_x(t)\, dt = o(1)$$

uniformly in x as $n \to \infty$, and hence, $J_n^{(k)}(x) = o(1)$ as $n \to \infty$ uniformly in x.

Now, if we use an argument similar to that in Bari [1, Vol. I, §32]

$$H_n^{(k)}(x) = -\int_{\pi/n < |t| < \pi} \frac{\cos(k + \frac{1}{2})t}{2 \sin \frac{1}{2} t} \varphi_x(-t)\, dt$$

$$= \int_{\pi/n < |t| < \pi} \frac{\cos(k + \frac{1}{2})t}{2 \sin \frac{1}{2} t} \varphi_x(t)\, dt$$

$$= \int_{\pi/n < |t| < \pi} \frac{\cos kt}{t} \varphi_x(t)\, dt + o(1)$$

uniformly in x as $n \to \infty$. Letting

$$H_{n,+}^{(k)}(x) = \int_{\pi/n}^{\pi} \frac{\cos kt}{t} \varphi_x(t)\, dt,$$

we may use an argument similar to that used for $J_{n,+}^{(k)}(x)$ to show $H_{n,+}^{(k)}(x) = o(1)$ and $H_{n,-}^{(k)}(x) = o(1)$ uniformly in x. Therefore,

$$|f(x) - S_n(f; x)| = \frac{1}{\pi} \frac{1}{(n + 1)^r} \cdot o(1) + o(1) \sum_{k=n+1}^{\infty} (k^{-r} - (k + 1)^{-r})$$

yielding $\|f - S_n(f)\|_c = o(n^{-r})$.

1.2 THEOREM. *If* $f^{(r)} \in \mathcal{V}_\Phi^*$, $0 < r < \infty$, $\Sigma\Phi^*(1/k) < \infty$, *then*

$$\|f - S_n(f)\|_C = o(n^{-r}) + O\left(n^{-r}\|f^{(r)}\|_{\Phi v}^*\right)$$

where

$$\|g\|_{\Phi v}^* = \lim_{\delta \to 0+} \sup\left\{\sum (g(x_i) - g(x_{i-1}))b_i : |Q| < \delta,\ \sum \Phi^*(b_i) < 1\right\}.$$

PROOF. For $g \in \mathcal{V}_\Phi^*$ we define

$$v(\delta, g) = \sup\left\{\sum (g(x_i) - g(x_{i-1}))b_i : |Q| < \delta,\ \sum \Phi^*(b_i) < 1\right\}$$

where the Q are partitions of a period. Then

$$\|g\|_{\Phi v}^* = \lim_{\delta \to 0+} v(\delta, g) \quad \text{and} \quad \omega(\delta, g) < cv(\delta, g)$$

where $c = 1$ if $\Phi^*(1) < 1$, $c = 1/\Phi^*(1)$ if $\Phi^*(1) > 1$.

Since the proof is similar to that of Theorem 1.1, we shall simply state where it differs.

Equation (2) becomes

(2)' $$\frac{1}{\pi} \int_0^{\pi/n} D_n^{(r)}(t)\varphi_x(-t)\, dt = O\left(v(\pi/n, f^{(r)})n^{-r}\right).$$

We now want to show

(5)' $$\int_{\pi/n < |t| < \pi} A_k^{(r)}(t)\varphi_x(-t)\, dt = O\left(v(\pi/n; f^{(r)})\right) + o(1)$$

$$= O\left(\|f^{(r)}\|_{\Phi v}^*\right) + o(1).$$

Equations (6) and (7) remain the same. For (∗) however, we do not use the mean value theorem, and will assume from here that the sums we work with will be integrated over $(\pi, 2\pi)$. We say again assume that the number of terms in the summation is even, since

$$\left| \int \sin u \varphi_x \left(\frac{u + k\pi}{k} \right) \frac{1}{u + k\pi} \right| < C \frac{1}{k} = o(1) \quad \text{as } n \to \infty.$$

Then equation (8) becomes

$$(8)' \quad \left| \varphi_x \left(\frac{u + (2h + 1)\pi}{k} \right) \left(\frac{1}{u + 2h\pi} - \frac{1}{u + (2h + 1)\pi} \right) \right| < \frac{v\left(\dfrac{(2h + 3)\pi}{k} ; f^{(r)} \right)}{(2h + 1)(2h + 2)\pi},$$

choosing M as before, the sum of the first M of these terms is

$$< cv\left(cM/k + o(1); f^{(r)} \right) = cv(o(1); f^{(r)})$$

as $n \to \infty$. The sum of the remaining terms is

$$< \frac{c \sup |f^{(r)}|}{M + [k/n]} = o(1)$$

as $n \to \infty$. If we consider

$$(9) \quad \left| \frac{\varphi_x \left(\dfrac{u + 2h\pi}{k} \right) - \varphi_x \left(\dfrac{u + (2h + 1)\pi}{k} \right)}{u + 2h\pi} \right|$$

and replace $u + 2h\pi$ in the denominator with $(2h + 2)\pi$, there is an error

$$(10)' \quad < \frac{v(\pi/k; f^{(r)})}{(2h + 1)(2h + 2)\pi} < \frac{v(\pi/n; f^{(r)})}{(2h + 1)(2h + 2)\pi}.$$

Thus, replacement gives an error in the sum of these terms which is less than $cv(\pi/n; f^{(r)})$.

Equations (11) and (12) remain the same, but (13) becomes, with the use of Hölder's inequality

$$(13)' \quad \frac{1}{2\pi} \sum_{h = ([k/n] - 1)/2}^{[(k-2)/2]} \frac{\Delta_k^h}{h + 1} < \frac{1}{2\pi} \sum_{h = 0}^{[(k-2)/2]} \frac{\Delta_k^h}{h + 1}$$

$$< v\left(\frac{\pi}{k} ; f^{(r)} \right) \left\| \left\{ \frac{1}{h + 1} \right\}_{h=0}^{[(k-2)/2]} \right\|_{\Phi^*}^0 < cv(\pi/k; f^{(r)}),$$

where

$$\left\| \{a_j\}_{j-1}^p \right\|_{\Phi}^0 = \sup \left\{ \sum_{i=1}^p a_i b_i : \sum_{i=1}^p \Phi^*(b_i) \le 1 \right\}.$$

A short digression seems to be in order to justify this last step.

Consider $\mathbf{R}^p$ and let $a = (a_1, \ldots, a_p)$, $b = (b_1, \ldots, b_p) \in \mathbf{R}^p$, and define

$$\|a\|_\Phi^0 = \sup\left\{ \sum_{i=1}^p a_i b_i \colon \sum \Phi^*(b_i) \leqslant 1 \right\}.$$

For any $a, b \in \mathbf{R}^p$,

$$\sum_{i=1}^p a_i b_i \leqslant \|a\|_\Phi^0 \|b\|_{\Phi^*}^0.$$

Thus, let $p = [(k-2)/2] + 1$.

Let

$$a = \left\{ a_{h+1} = \Delta_k^h \right\}_{h=0}^{p-1}, \qquad b^p = \left\{ b_i = 1/(i+1) \right\}_{i=0}^{p-1}.$$

Then

$$\sum_{i=0}^{[(k-2)/2]} \frac{\Delta_k^i}{(i+1)} \leqslant \|a\|_\Phi^0 \|b^p\|_{\Phi^*}^0.$$

Now, since $\sum_{i=0}^\infty \Phi^*(1/(i+1)) < \infty$,

$$\lim_{p \to \infty} \|b^p\|_{\Phi^*}^0 = A < \infty,$$

and

$$\|a\|_\Phi^0 = \sup\left\{ \sum_{h=0}^p b_i \left(f^{(r)}\!\left(x + \frac{\theta}{k} + \frac{2h\pi}{k} \right) \right.\right.$$
$$\left.\left. - f^{(r)}\!\left(x + \frac{\theta}{k} + \frac{(2h+1)\pi}{k} \right) \right) \colon \sum \Phi^*(b_i) < 1 \right\}$$

$$< v\!\left(\pi/k; f^{(r)} \right).$$

Putting these two results together yields

$$\sum_{i=0}^{[(k-2)/2]} \frac{\Delta_k^i}{(i+1)} \leqslant A v(\pi/k; f^{(r)}).$$

Thus,

$$\left| L_n^{(k)}(x) \right| \leqslant \int_\pi^{2\pi} \sin u \left(cv(\pi/n, f^{(r)}) + o(1) \right) du$$

$$\leqslant cv\!\left(\pi/n; f^{(r)} \right) + o(1),$$

and we can show that

$$\int_{\pi/n < |t| < \pi} A_k^{(r)}(t) \varphi_x(-t)\, dt = O\left(\|f^{(r)}\|_{\Phi v}^* \right) + o(1),$$

and the conclusion follows from equation (3).

2. **Approximation in the Orlicz space norm.** The theorem which we shall prove in this section is a generalization of the fundamental result of S. M. Nikol'skiĭ [8] and B. I. Golubov [3].

THEOREM. *Let* $f \in \mathcal{V}_{\Phi}^{(r)}$, Φ *an N-function,* $r \geqslant 0$. *Further suppose* $\{x_i\}$ *are the points of* $[-\pi, \pi)$ *at which* $f^{(r)}$ *is discontinuous. Let*

$$\sigma_i = f^{(r)}(x + 0) - f^{(r)}(x - 0).$$

Then, for each reflexive Δ' *N-function,* Ψ, *with* $\int_1^{\infty} \Psi(t^{-1})\, dt < \infty$ *and*

$$\lim_{x \to 0} \Psi(x)/\Phi(x) = 0,$$

we have, for $f^{(r)}$ *continuous,*

$$\|f - S_n(f)\|_{\Psi} = o\big(1/n^r \Psi^{-1}(n)\big),$$

otherwise,

$$\|f - S_n(f)\|_{\Psi} \leqslant \frac{1}{n^r \Psi^{-1}(n)} \left[o(1) + 1 \middle/ \Psi^{-1}\left(1 \middle/ \frac{C_{\Psi}^2 S}{2} \sum_{j=1}^{\infty} \Psi(\sigma_j) \right) \right].$$

To prove Theorem 2.1 we must first develop several other results.

Let α be an arbitrary real number, n any nonnegative integer, and r any nonnegative real number. Then we define

$$\varphi_n(\alpha, r, x) = \sum_{k=n+1}^{\infty} \frac{\cos(kx + \alpha)}{k^{r+1}}.$$

2.1 LEMMA. *For* Φ *a* Δ' *N-function such that* $\int_{|t|>1} \Phi(t^{-1}) < \infty$, *and for* $r \geqslant 0$,

(1) $$\varphi_n(\alpha, r, x) = \frac{1}{\Gamma(r+1)n^r} \varphi_*(\alpha, r, nx) + \varepsilon_n(\alpha, r, x)$$

where

(2) $$\varphi_*(\alpha, r, t) = \int_0^{\infty} e^{-v} v^r \frac{(v \cos(t + \alpha) - t \sin(t + \alpha))}{v^2 + t^2}\, dv$$

and

$$\int_{-\pi}^{\pi} \Phi\big(n^r \Phi^{-1}(n)\varepsilon_n(\alpha, r, x)\big)\, dx = O\big(\Phi(\Phi^{-1}(n)/n)\big).$$

To prove Lemma 2.1 we first need several auxiliary results.

2.1A LEMMA. *For* $v < n$, $|t| < n\pi$,

$$(3) \quad \left| \frac{1}{\frac{t^2}{n^2}\exp\!\left(\frac{-v}{n+1}\right) + \left(1 - \exp\!\left(\frac{-v}{n+1}\right)\right)^2} - \frac{1}{\frac{v^2 + t^2}{n^2}} \right|$$

$$= O\!\left(\frac{1+v}{n}\right) \Big/ \frac{v^2 + t^2}{n^2}.$$

PROOF. The left-hand side of (3) equals

$$\left| \frac{\frac{v^2}{n^2} + \frac{t^2}{n^2}\left(1 - \exp\!\left(\frac{-v}{n+1}\right)\right) - \left(1 - \exp\!\left(\frac{-v}{n+1}\right)\right)^2}{\left(\left(1 - \exp\!\left(\frac{-v}{n+1}\right)\right)^2 + \frac{t^2}{n^2}\exp\!\left(\frac{-v}{n+1}\right)\right)\left(\frac{v^2 + t^2}{n^2}\right)} \right|$$

If we use the substitution

$$1 - \exp\!\left(\frac{-v}{n+1}\right) = \frac{v}{n+1} + O\!\left(\frac{v^2}{n^2}\right) = \frac{v}{n} + O\!\left(\frac{v + v^2}{n^2}\right)$$

in the above equation, the numerator becomes

$$\left| \frac{v^2}{n^2} + \frac{t^2}{n^2}\left(1 - \exp\!\left(\frac{-v}{n+1}\right)\right) - \left(1 - \exp\!\left(\frac{-v}{n+1}\right)\right)^2 \right|$$

$$= \left| \frac{v^2}{n^2} + \frac{t^2}{n^2}\left(\frac{v}{n} + O\!\left(\frac{v+v^2}{n^2}\right)\right) - \frac{v^2}{n^2} \right.$$

$$\left. + O\!\left(\frac{v^2 + v^3}{n^3}\right) + O\!\left(\frac{v^2 + 2v^3 + v^4}{n^4}\right) \right|$$

$$= O\!\left(\frac{v^2}{n^3}\right) + O\!\left(\frac{vt^2}{n^3}\right) + O\!\left(\frac{v^3}{n^3}\right) = O\!\left(\frac{v^2 + t^2 v + v^3}{n^3}\right).$$

On the other hand,

$$1 - \exp\!\left(\frac{-v}{n+1}\right) > \frac{v}{n+1} > \frac{v}{2n}$$

and

$$\frac{t^2}{n^2}\exp\!\left(\frac{-v}{n+1}\right) > \frac{t^2}{n^2}\exp(-1).$$

Using the above approximations, the left side of (3) is

$$< \frac{O\!\left(\frac{v^2 + t^2 v + v^3}{n^3}\right)}{\left(\frac{v^2}{4n^2} + e^{-1}\frac{t^2}{n^2}\right)\left(\frac{v^2 + t^2}{n^2}\right)}$$

$$< \frac{O\!\left((v^2 + t^2 v + v^3)/n\right)}{(v^2 + t^2)^2/n^2} < \frac{O\!\left((1+v)/n\right)}{(v^2 + t^2)/n^2}.$$

2.1B LEMMA. *If* $0 < \rho < 1$, $|x| < \pi$, *then*

(4) $$\left| \frac{1}{(1 - \rho)^2 + 4\rho \sin^2 x/2} - \frac{1}{(1 - \rho)^2 + \rho x^2} \right| = \frac{O(x^2)}{(1 - \rho)^2 + \rho x^2}.$$

PROOF. The left side of (4) equals

$$\left| \frac{\rho x^2 \left(1 + \frac{2 \sin x/2}{x}\right)\left(1 - \frac{2 \sin x/2}{x}\right)}{((1 - \rho)^2 + 4\rho \sin^2 x/2)(1 - \rho)^2 + \rho x^2} \right|.$$

Now, for $x \neq 0$,

$$\left(1 + \frac{2}{x} \sin \frac{x}{2}\right) < c \quad \text{and} \quad \left(1 - \frac{2}{x\cdot} \sin \frac{x}{2}\right) = O(x^2).$$

Hence, the left side of (4) equals

$$\left| \frac{\rho O(x^4)}{((1 - \rho)^2 + 4\rho \sin^2 x/2)((1 - \rho)^2 + \rho x^2)} \right| < \frac{O(x^2)}{(1 - \rho)^2 + \rho x^2}.$$

2.1C LEMMA. *When* $0 < \rho < 1$ *and* $|x| < \pi$,

$$\cos((n + 1)x + \alpha) - \rho \cos(nx + \alpha)$$

$$= (1 - \rho)\cos(nx + \alpha) - x \sin(nx + \alpha) + O(x^2).$$

PROOF.

$$\cos((n + 1)x + \alpha) - \rho \cos(nx + \alpha)$$

$$= \cos x \cos(nx + \alpha) - \rho \cos(nx + \alpha) - \sin x \sin(nx + \alpha)$$

$$= \cos(nx + \alpha) + O(x^2)\cos(nx + \alpha) - \rho \cos(nx + \alpha)$$

$$- x \sin(nx + \alpha) + O(x^3)\sin(nx + \alpha)$$

$$= (1 - \rho)\cos(nx + \alpha) - x \sin(nx + \alpha) + O(x^2).$$

2.1D LEMMA. *Let* Φ *be a* Δ_2 *N-function such that* $\int_{|t|>1}\Phi(t^{-1})\, dt < \infty$. *Define for* $r > 0$,

$$h_r(t) = \int_0^\infty \frac{e^{-v}v^r}{v^2 + t^2}\, dv$$

and

$$g_r(t) = t \int_0^\infty \frac{e^{-v}v^r}{v^2 + t^2}\, dv = th_r(t).$$

Then there exist constants $C = C(\Phi, r)$ *and* $B = B(\Phi, r)$ *such that for* $r > 1$

$$\int_{-\infty}^\infty \Phi(h_r(t))\, dt < C,$$

and for $r > 0$

$$\int_{-\infty}^{\infty} \Phi(g_r(t))\, dt \leqslant B.$$

PROOF. We have that for $0 < t < \infty$, $r > 0$,

$$h_r(t) < \frac{1}{t^2}\int_0^\infty e^{-v}v^r\, dv < \frac{C(r)}{t^2} = \frac{C}{t^2}$$

and for $0 < t < 1$,

$$h_r(t) = \int_0^\infty \frac{e^{-ut}(ut)^r t\, du}{t^2(1+u^2)}$$

$$< \frac{1}{t^2}\int_0^1 e^{-ut}(ut)^r t\, du + \int_1^\infty e^{-ut}(ut)^{r-2}t\, du$$

$$= \frac{1}{t^2}\int_0^t e^{-v}v^r\, dv + \int_t^1 e^{-v}v^{r-2}\, dv + \int_1^\infty e^{-v}v^{r-2}\, dv$$

$$< \frac{1}{t^2}\int_0^t v^r\, dv + \int_t^1 v^{r-2}\, dv + C$$

$$\leqslant \begin{cases} C(t^{r-1}+1), & r \neq 1, r > 0, \\ C(1+|\log t|), & r = 1. \end{cases}$$

Since $\Phi(c/t^2) = O(c/t^2) = O(1/t^2)$ as $t \to \infty$, we have

$$\int_{|t|>1} \Phi(h_r(t))\, dt < \int_{|t|>1}\Phi\left(\frac{c}{t^2}\right) dt = \int_{|t|>1} O\left(\frac{1}{t^2}\right) dt = I_1 < \infty.$$

If $r > 1$, we have

$$\int_{|t|<1} \Phi(h_r(t))\, dt \leqslant \int_{|t|<1}\Phi(c(1+|t|^{r-1}))\, dt \leqslant I_2 < \infty.$$

If $r = 1$, we have

$$\int_{|t|<1}\Phi(h_r(t))\, dt \leqslant \int_{|t|<1}\Phi(c+c|\log|t||)\, dt \leqslant c + \frac{1}{2}\int_{|t|<1}\Phi(2c\log|t|)\, dt.$$

Now since Φ is Δ_2 for large values we know that there exists u_0 and $\beta > 1$ such that $u \geqslant u_0$ implies $\Phi(u) < A(u_0, \Phi)u^\beta$. Let $\alpha = \beta + 1$. Then $|\log|t|| = o(t^{-1/\alpha})$ as $t \to 0$. Choose $2 > \varepsilon > 0$. Then there exists $\delta = \delta(u_0, \varepsilon)$, $\delta < u_0^{-\alpha}$, such that for $|t| < \delta$,

$$2ct^{1/\alpha}|\log t| < \varepsilon/2,$$

and

$$\int_{|t|<1} \Phi(2c\,\log|t|)\,dt \leqslant \int_{\delta<|t|<1} \Phi(2c\,\log|t|)\,dt + \frac{\varepsilon}{2} \int_{|t|<\delta} \Phi(t^{-1/\alpha})\,dt$$

$$\leqslant A_1 + \frac{\varepsilon}{2} \int_{u>\delta^{-1/\alpha}} \alpha\Phi(u)/u^{(1+\alpha)}\,du$$

$$= A_1 + \frac{\varepsilon}{2} \int_{|u|>\delta^{-1/\alpha}} \alpha\Phi(u)u^{-\beta-2}\,du$$

$$\leqslant A_1 + \frac{\varepsilon\alpha A}{2} \int_{|u|>\delta^{-1/\alpha}} u^{-2}\,du = I_2 < \infty.$$

Thus, we have that

$$\int_{-\infty}^{\infty} \Phi(h_r(t))\,dt < \infty \quad \text{for } r > 1.$$

Now, suppose $r > 0$, then for $r \neq 1$,

$$\int_{0<|t|<1} \Phi(g_r(t))\,dt \leqslant \int_{0<|t|<1} \Phi(c(t+t^r))\,dt < \infty,$$

and for $r = 1$,

$$\int_{0<|t|<1} \Phi(g_r(t))\,dt \leqslant \frac{1}{2} \int_{0<|t|<1} \Phi(2ct)\,dt + \frac{1}{2} \int_{0<|t|<1} \Phi(2ct\,\log|t|)\,dt$$

since

$$\lim_{t\to 0} |t\,\log|t|| = 0,$$

$|t|\,|\log|t||$ is bounded on $(-1, 1)$. Hence, $\Phi(2ct\,\log|t|)$ is bounded on $(-1, 1)$ and has a finite integral. Thus,

$$\int_{0<|t|<1} \Phi(g_r(t))\,dt < \infty, \qquad r > 0.$$

For all $r \geqslant 0$, we have, by hypothesis,

$$\int_{|t|>1} \Phi(g_r(t))\,dt \leqslant \int_{|t|>1} \Phi(c/t)\,dt < \infty.$$

Thus, for $r > 0$,

$$\int_{-\infty}^{\infty} \Phi(g_r(t))\,dt < \infty.$$

We now return to the *proof of Lemma* 2.1. From Nikol'skiĭ [8] we have that

$$\varphi_n(\alpha, r, x)$$

(5)
$$= \frac{1}{\Gamma(r+1)} \int_0^1 \rho^n \left(\log \frac{1}{\rho}\right)^r \frac{\cos((n+1)x+\alpha) - \rho\cos(nx+\alpha)}{\rho^2 - 2\rho\cos x + 1}\,d\rho.$$

By Lemmas 2.1B and 2.1C,

$$\frac{\cos((n + 1)x + \alpha) - \rho \cos(nx + \alpha)}{\rho^2 - 2\rho \cos x + 1}$$

$$= \frac{\cos(nx + \alpha)(1 - \rho) - x \sin(nx + \alpha)}{(1 - \rho)^2 + \rho x^2} + \frac{O(x^2)}{(1 - \rho)^2 + \rho x^2}.$$

We may write

$\varphi_n(\alpha, r, x)$

$$= \frac{1}{\Gamma(r + 1)} \int_0^1 \rho^n \left(\log \frac{1}{\rho}\right)^r \frac{\cos(nx + \alpha)(1 - \rho) - x \sin(nx + \alpha)}{(1 - \rho)^2 + \rho x^2} \, d\rho$$

$$+ \delta_n(r, x)$$

$$\equiv \theta_n(\alpha, r, x) + \delta_n(r, x),$$

and

$$|\delta_n(r, x)| = \left| O(x^2) \int_0^1 \frac{\rho^n (\log 1/\rho)^r}{(1 - \rho)^2 + \rho x^2} \, d\rho \right|$$

$$< \left| O(1) \int_0^1 \rho^{n-1} (\log 1/\rho)^r \, d\rho \right|$$

$$= O(n^{-r-1}).$$

Now consider $\theta_n(\alpha, r, x)$. We substitute $\rho = \exp(-v/(n + 1))$ and $t = nx$ in the expression defining θ_n, obtaining

$$\theta_n\left(\alpha, r, \frac{t}{n}\right) = \frac{1}{\Gamma(r + 1)n^{r+1}} \int_0^\infty e^{-v}v^r \frac{(1 - e^{-v/(n+1)})\cos(t + \alpha) - t/n \sin(t + \alpha)}{(1 - e^{-v/(n+1)})^2 + \frac{t^2}{n^2} e^{-v/(n+1)}} \, dv.$$

Now let

$$\theta_n(\alpha, r, t/n) = \frac{1}{\Gamma(r + 1)(n + 1)^{r+1}} \int_0^0 \cdots dv + \tilde{\delta}_n(\alpha, r, t)$$

where

$$|\tilde{\delta}_n(\alpha, r, t)| < \frac{1}{(n + 1)^{r+1}\Gamma(r + 1)} \cdot \frac{\pi + 1}{(1 - e^{-n/(n+1)})^2} \int_n^\infty e^{-v}v^r \, dv$$

$$< c \exp(-n/2)$$

where c depends only on r.

Using Lemma 2.1A and

$$1 - \exp\left(\frac{-v}{n + 1}\right) = \frac{v}{n} + O\left(\frac{v + v^2}{n^2}\right)$$

we have

$$\theta_n(\alpha, r, t/n)$$

$$= \tilde{\delta}_n(\alpha, r, t) + \frac{1}{\Gamma(r+1)(n+1)^{r+1}}$$

$$\cdot \int_0^n \frac{e^{-v}v^r}{n} \left[\frac{v\cos(t+\alpha) - t\sin(t+\alpha) + O\left(\frac{v+v^2}{n}\right) - \left(1 + O\left(\frac{1+v}{n}\right)\right)}{\frac{v^2+t^2}{n^2}} \right] dx$$

$$= \tilde{\delta}_n(\alpha, r, t) + \frac{1}{\Gamma(r+1)(n+1)^{r+1}} \int_0^n e^{-v} \frac{v^r O((1+v)|t|)}{v^2+t^2} \, dv$$

$$+ \frac{1}{\Gamma(r+1)(n+1)^{r+1}} \int_0^n e^{-v} \frac{v^r O(v+v^2)}{v^2+t^2} \, dv$$

$$- \frac{1}{\Gamma(r+1)n^r} \int_n^\infty e^{-v} \frac{v^r(v\cos(t+\alpha) - t\sin(t+\alpha))}{v^2+t^2} \, dv$$

$$+ \left(\frac{n}{(n+1)^{r+1}} - \frac{1}{n^r} \right) \frac{1}{\Gamma(r+1)} \int_0^n e^{-v} \frac{v^r(v\cos(t+\alpha) - t\sin(t+\alpha))}{v^2+t^2} \, dv$$

$$+ \frac{1}{n^r} \frac{1}{\Gamma(r+1)} \int_0^\infty e^{-v} \frac{v^r(v\cos(t+\alpha) - t\sin(t+\alpha))}{v^2+t^2} \, dv$$

$$= \tilde{\delta}_n(\alpha, r, t) + I_r(t) + II_r(t) - III_r(t) + IV_r(t) + \varphi_*(\alpha, r, t)$$

$$= \varepsilon'_n(\alpha, r, t) + \varphi_*(\alpha, r, t).$$

Using Lemma 2.1D, we have that,

$$|IV_r(t)| \leqslant \left| \frac{n}{(n+1)^{r+1}} - \frac{1}{n^r} \right| \left| \frac{1}{\Gamma(r+1)} \right|$$

$$\cdot |h_{r+1}(t)\cos(t+\alpha) - g_r(t)\sin(t+\alpha)|$$

so

$$\int_{-n\pi}^{n\pi} \Phi\left(n^r \Phi^{-1}(n) IV_r(t)\right) \leqslant c_{\Phi, r} \Phi(\Phi^{-1}(n)/n) \int_0^{n\pi} \Phi\left(|h_{r+1}(t)| + |g_r(t)|\right) dt$$

$$\leqslant c\Phi(\Phi^{-1}(n)/n) = O\left(\Phi(\Phi^{-1}(n)/n)\right).$$

We note that

$$O(v+v^2) = \begin{cases} O(v^2), & v \geqslant 1, \\ O(v), & v \leqslant 1. \end{cases}$$

We first consider

$$|\text{III}_r(t)| \leqslant \frac{1}{n^r\Gamma(r+1)} \int_n^\infty \frac{e^{-v}v^r(2v)}{v^2}\, dv = O(e^{-n/2}).$$

Now, suppose $r > 0$. Then

$$\left|(n+1)^{r+1}\Gamma(r+1)\text{I}_r(t)\right| = \int_0^n \frac{e^{-v}v^rO((1+v)|t|)}{v^2+t^2}\, dv$$

$$= \left[\int_{0<|t|<v<n} + \int_{\substack{0<v<|t| \\ v<n}}\right] e^{-v}\frac{v^rO((1+v)|t|)}{v^2+t^2}\, dv.$$

Let the quantity represented by the first integral on the right be A and the second be B. Then,

$$A \leqslant \int_{0<|t|<v<n} e^{-v}\frac{\left(O(v^{r+1}) + O(v^{r+2})\right)}{v^2+t^2}\, .dv$$

$$\leqslant \int_0^n O(e^{-v}v^{r-1})\, dv + \int_0^n O(e^{-v}v^r)\, dv \leqslant C.$$

$$B \leqslant \int_0^{|t|} e^{-v}\frac{v^rO((1+v)|t|)}{v^2+t^2}\, dv \leqslant \int_0^{|t|} e^{-v}\frac{v^rO(1+v)}{|t|}\, dv$$

$$\leqslant \int_0^{|t|} e^{-v}\frac{v^rO(1+v)}{v}\, dv = \int_0^{|t|} e^{-v}v^{r-1}O(1+v)\, dv \leqslant C.$$

The constants, C, in A and B depend only on r. Hence

$$|\text{I}_r(t)| = O\left((n+1)^{-r-1}\right).$$

Also,

$$\left|(n+1)^{r+1}\Gamma(r+1)\text{II}_r(t)\right| = \int_0^1 \frac{e^{-v}v^rO(v)}{v^2+t^2}\, dv + \int_1^n \frac{e^{-v}v^rO(v^2)}{v^2+t^2}\, dv$$

$$\leqslant C + \int_1^n e^{-v}O(v^2)\, dv = O(1).$$

Thus, $|\text{II}_r(t)| = O((n+1)^{-r-1})$. Hence,

$$\int_{-n\pi}^{n\pi} \Phi\left(n^r\Phi^{-1}(n)\varepsilon_n'(\alpha, r, t)\right) dt$$

$$\leqslant \frac{1}{2}\int_{-n\pi}^{n\pi} \Phi\left(2n^r\Phi^{-1}(n)(\bar\delta + \text{I}_r + \text{II}_r - \text{III}_r)\right) dt$$

$$+ \frac{1}{2}\int_{-n\pi}^{n\pi} \Phi\left(2n^r\Phi^{-1}(n)\text{IV}_r(t)\right) dt$$

$$= O\left(n\Phi(\Phi^{-1}(n)/n)\right) + O\left(\Phi(\Phi^{-1}(n)/n)\right).$$

We now suppose $r = 0$. Then

$$|(n + 1)\text{II}_0(t)| = \int_0^1 e^{-v} \frac{O(v + v^2)}{v^2 + t^2} \, dv + \int_1^n e^{-v} \frac{O(v + v^2)}{v^2 + t^2} \, dv$$

$$\leqslant \int_0^1 \frac{e^{-v} O(v + v^2)}{v^2 + t^2} \, dv + O(1).$$

For $|t| \geqslant 1$,

$$\int_0^1 e^{-v} \frac{O(v + v^2)}{v^2 + t^2} \, dv \leqslant \int_0^1 \frac{e^{-v} O(v) \, dv}{t^2} \leqslant \int_0^1 e^{-v} O(v) \, dv = O(1),$$

and for $0 < |t| < 1$,

$$\int_0^1 e^{-v} \frac{O(v + v^2)}{v^2 + t^2} \, dv \leqslant \int_0^1 \frac{O(v)}{v^2 + t^2} \, dv$$

$$= O\left(\log\left(\frac{1 + t^2}{t^2}\right)\right) = O\left(\log 2 + \log \frac{1}{t}\right).$$

Thus,

$$|(n + 1)\text{II}_0(t)| = O(1) + \begin{cases} 0, & |t| \geqslant 1, \\ O(\log 1/t), & |t| < 1. \end{cases}$$

Now,

$$|(n + 1)\text{I}_0(t)| = \int_0^1 \frac{e^{-v} O((1 + v)|t|)}{v^2 + t^2} \, dv + \int_1^n e^{-v} \frac{O((1 + v)|t|)}{v^2 + t^2} \, dv,$$

where

$$\int_1^n \frac{e^{-v} O((1 + v)|t|)}{v^2 + t^2} \, dv \leqslant \int_1^n \frac{e^{-v} O(v^2)}{v^2} + \int_1^n \frac{e^{-v} O(t^2)}{t^2} \, dv = O(1),$$

and, if $|t| \geqslant 1$,

$$\int_0^1 e^{-v} \frac{O((1 + v)|t|)}{v^2 + t^2} \, dv \leqslant \int_0^{|t|} \frac{e^{-v} O(t^2)}{t^2} \, dv = O(1),$$

while for $|t| < 1$,

$$\int_0^1 e^{-v} \frac{O((1+v)|t|)}{v^2 + t^2} \, dv \ll \int_0^{|t|} \frac{e^{-v} O(|t|)}{t^2} \, dv + \int_{|t|}^1 \frac{e^{-v} O(|t|)}{v^2 + t^2} \, dv$$

$$+ \int_{|t|}^1 \frac{e^{-v} O(v|t|)}{v^2 + t^2} \, dv$$

$$\ll O(1) + \int_{|t|}^1 \frac{O(v)}{v^2 + t^2} \, dv + \int_{|t|}^1 \frac{e^{-v} O(v^2)}{v^2} \, dv$$

$$= O(1) + O\left(\log\left(\frac{1 + t^2}{2t^2} \right) \right).$$

Thus,

$$|(n+1)\mathrm{I}_0(t)| = O(1) + \begin{cases} 0, & |t| \geqslant 1, \\ O(\log|1/t|), & |t| < 1. \end{cases}$$

The above conditions show, if $\varepsilon_r^*(\alpha, 0, t) = \tilde{\delta}(\alpha, 0, t) + \mathrm{I}_0(t) + \mathrm{II}_0(t) - \mathrm{III}_0(t)$, then

$$\varepsilon_r^*(\alpha, 0, t) = O\left(\frac{1}{n+1} \right) + \begin{cases} 0, & |t| \geqslant 1, \\ O(\log|1/t|/(n+1)), & |t| < 1, \end{cases}$$

and, hence

$$\int_{-n\pi}^{n\pi} \Phi\big(\Phi^{-1}(n)\varepsilon_n^*(\alpha, 0, t)\big) \, dt$$

$$= 2\int_0^1 \Phi\left(\frac{\Phi^{-1}(n)}{n+1} O\left(1 + \log \frac{1}{t} \right) \right) dt + \int_{1 < |t| < n\pi} \Phi\left(\frac{\Phi^{-1}(n)}{n+1} \right) dt.$$

Substituting $t = e^{-w}$ in the first integral gives

$$\int_0^1 \Phi\left(\frac{\Phi^{-1}(n)}{n+1} O\left(1 + \log \frac{1}{t} \right) \right) dt = \int_0^\infty \Phi\left(\frac{\Phi^{-1}(n)}{n+1} \cdot O(1 + w) \right) e^{-w} \, dw$$

$$\ll c\Phi\left(\frac{\Phi^{-1}(n)}{n+1} \right) \int_0^\infty e^{-w} \, dw + \int_0^\infty \Phi\left(\frac{\Phi^{-1}(n)}{n+1} O(w) \right) e^{-w} \, dw$$

$$\ll c\Phi\left(\frac{\Phi^{-1}(n)}{n+1} \right) + c\Phi\left(\frac{\Phi^{-1}(n)}{n+1} \right) \int_0^\infty \Phi(w) e^{-w} \, dw.$$

Now Φ is Δ_2 implies that there exists u_0 such that for $w \geqslant u_0$, $\Phi(w) < Aw^p$ for some $p > 1$, A a constant. Applying this,

$$\int_0^\infty \Phi(w) e^{-w} \, dw \ll \int_0^{u_0} \Phi(w) e^{-w} \, dw + \int_{u_0}^\infty Aw^p e^{-w} \, dw < \infty.$$

Hence,

$$\int_{-n\pi}^{n\pi} \Phi\big(\Phi^{-1}(n)\varepsilon_n^*(\alpha, 0, t)\big)\, dt < O\big(n\Phi(\Phi^{-1}(n)/n)\big).$$

Then, since $\varepsilon_n'(\alpha, 0, t) = \varepsilon_n^*(\alpha, 0, t) + \mathrm{IV}_0(t)$,

$$\int_{-n\pi}^{n\pi} \Phi\big(\Phi^{-1}(n)\varepsilon_n'(\alpha, 0, t)\big)\, dt \leqslant nO\big(\Phi(\Phi^{-1}(n)/n)\big) + O\big(\Phi(\Phi^{-1}(n)/n)\big).$$

Define $\varepsilon_n(\alpha, r, x) = \varepsilon_n'(\alpha, r, nx) + \delta_n(r, x)$, for $|x| < \pi$. Then,

$$\int_{-\pi}^{\pi} \Phi\big(n^r\Phi^{-1}(n)\varepsilon_n(\alpha, n, x)\big)\, dx$$

$$= \int_{-n\pi}^{n\pi} \Phi\left(n^r\Phi^{-1}(n)\left(\varepsilon_n'(\alpha, r, t) + \delta_n\left(r, \frac{t}{n}\right)\right)\right) \frac{dt}{n}$$

$$= O\big(\Phi(\Phi^{-1}(n)/n)\big).$$

Hence,

$$\varphi_n(\alpha, r, x) = \frac{1}{\Gamma(r+1)n^r} \int_0^\infty e^{-v}v^r\, \frac{(v\cos(nx+\alpha) - (nx)\sin(nx+\alpha))}{v^2 + (nx)^2}\, dv$$

$$+ \varepsilon_n(\alpha, r, x)$$

$$= \varphi_*(\alpha, r, nx) + \varepsilon_n(\alpha, r, x),$$

where

$$\int_{-\pi}^{\pi} \Phi\big(n^r\Phi^{-1}(n)\varepsilon_n(\alpha, r, x)\big)\, dx = O\big(\Phi(\Phi^{-1}(n)/n)\big).$$

2.2 LEMMA. *Let Φ be a Δ' N-function such that $\int_{|t|>1}\Phi(t^{-1})\, dt < \infty$, and let $r > 0$. Then there exists $S = S(\Phi, \alpha, r)$ such that*

$$\int_0^{2\pi} \Phi\big(n^r\Phi^{-1}(n)\varphi_n(\alpha, r, x)\big)\, dx < S + O\big(\Phi(\Phi^{-1}(n)/n)\big).$$

PROOF. By Lemma 2.1,

$$\int_{-\pi}^{\pi} \Phi\big(n^r\Phi^{-1}(n)\varphi_n(\alpha, r, x)\big)\, dx$$

$$< \frac{1}{2}\int_{-\pi}^{\pi} \Phi\left(\frac{2\Phi^{-1}(n)}{\Gamma(r+1)}\, \varphi_*(\alpha, r, nx)\right) dx + \frac{1}{2}\int_{-\pi}^{\pi} \Phi\big(2n^r\Phi^{-1}(n)\varepsilon_n(\alpha, r, x)\big)\, dx$$

$$< c\int_{-\pi}^{\pi} \Phi\big(\Phi^{-1}(n)\varphi_*(\alpha, r, nx)\big)\, dx + O\big(\Phi(\Phi^{-1}(n)/n)\big)$$

$$< c\int_{-n\pi}^{n\pi} \Phi\big(\varphi_*(\alpha, r, x)\big)\, dx + O\big(\Phi(\Phi^{-1}(n)/n)\big).$$

Now $\varphi_*(\alpha, r, x) = \cos(x + \alpha)h_{r+1}(x) - g_r(x)\sin(x + \alpha)$, and so

$$\Phi\big(|\varphi_*(\alpha, r, x)|\big) \leqslant c\big[\Phi(h_{r+1}(x)) + \Phi(g_r(x))\big]$$

and

$$\int_{-\infty}^{\infty} \Phi\big(\varphi_*(\alpha, r, x)\big)\, dx \leqslant c\left[\int_{-\infty}^{\infty} \Phi(h_{r+1}(x))\, dx + \int_{-\infty}^{\infty} \Phi(g_r(x))\, dx\right] < \infty,$$

by Lemma 2.1D, which yields the conclusion of the lemma with

$$S(\Phi, \alpha, r) = \int_{-\infty}^{\infty} \Phi\big(\varphi_*(\alpha, r, x)\big)\, dx.$$

2.3 LEMMA. *Let Φ be a Δ_2 N-function and let $\delta > 0, r \geqslant 0$. Then there exists a constant $C = C(r, \delta, \Phi)$ such that*

$$\int_{\delta < |x| < \pi} \Phi\big(n^r\Phi^{-1}(n)\varphi_n(\alpha, r, x)\big)\, dx = C\Phi\big(\Phi^{-1}(n)/n\big).$$

PROOF. From the proof of Lemma 2.1, equation (5), we have that

$$\varphi_n(\alpha, r, x) = \frac{1}{\Gamma(r + 1)}$$

$$\cdot \int_0^1 \rho^n (\log 1/\rho)^r \frac{(\cos((n + 1)x + \alpha) - \rho\cos(nx + \alpha))}{\rho^2 - 2\rho\cos x + 1}\, d\rho$$

$$\leqslant \frac{1}{\Gamma(r + 1)} \int_0^1 \frac{2\rho^{n-1}(\log 1/\rho)^r}{4\sin^2 x/2}\, d\rho.$$

For $|x| > \delta$,

$$\int_0^1 \frac{2\rho^{n-1}(\log 1/\rho)^r}{4\sin^2 x/2}\, d\rho \leqslant \frac{\pi^2}{4\delta^2} \int_0^1 \rho^{n-1}(\log 1/\rho)^r\, d\rho = \frac{\pi^2}{4\delta^2} \cdot \frac{\Gamma(r + 1)}{n^{r+1}}$$

so

$$\int_{\delta < |x| < \pi} \Phi\big(n^r\Phi^{-1}(n)\varphi_n(\alpha, r, x)\big)\, dx \leqslant \int_{\delta < |x| < \pi} \Phi\left(\frac{\pi^2}{4\delta^2} \cdot \frac{\Phi^{-1}(n)}{n}\right) dx$$

$$\leqslant C\Phi\big(\Phi^{-1}(n)/n\big).$$

We define

$$\varphi_{(0),0}(x) = \sum_{k=1}^{\infty} k^{-1}\sin kx = \frac{\pi - x}{2}, \qquad 0 < x < 2\pi.$$

Then,

$$\varphi_{(r),0}(x) = \int D^{(r)}(x - t)\varphi_{(0),0}(t)\, dt$$

$$= \sum_{k=1}^{\infty} k^{-r-1}\cos\left(kx - \frac{r+1}{2}\pi\right) = \varphi_0\left(-\frac{r+1}{2}\pi, r, x\right).$$

We note that $S[\varphi_{(r),0}] = \varphi_{(r),0}$, $r > 0$, so that we shall write

$$\varphi_{(r),0} - S_n(\varphi_{(r),0}) = \varphi_{(r),n}(x) = \sum_{k=n+1}^{\infty} \frac{\cos(kx - (r+1)\pi/2)}{k^{r+1}}$$

$$= \int D^{(r)}(x - t)\varphi_{(0),n}(t)\, dt.$$

2.4 LEMMA. *Let Φ be a Δ' N-function such that $\int_{|t|>1}\Phi(t^{-1})\, dt < \infty$ and let $r > 0$. Suppose $-\pi < x_1 < x_2 < \cdots < x_m < \pi$ and define*

$$f(x) = \sum_{k=1}^{m} \sigma_k \varphi_{(r),0}(x - x_k)$$

where $\sigma = \{\sigma_k\}$ is a sequence of real numbers. If

$$0 < \delta < \min\left\{\pi + \frac{x_1 - x_m}{2}, \frac{x_2 - x_1}{2}, \dots, \frac{x_m - x_{m-1}}{2}\right\},$$

then

$$\|f - S_n(f)\|_{\Phi} < \frac{2}{n^r\Phi^{-1}(n)} \cdot 1 \bigg/ \Phi^{-1}\left[\frac{1}{\frac{SC_{\Phi}^2}{2}\sum_{j=1}^{m}\Phi(\sigma_j) + O_{m,\delta,\sigma}\left(\Phi\left(\frac{\Phi^{-1}(n)}{n}\right)\right)}\right]$$

where $f(n) = O_{m,\delta,\sigma}(g(n))$ means $|f(n)/g(n)| \leq c$ where $c = c(m, \delta, \sigma)$.

PROOF.

$$\frac{1}{2}\left\|n^r\Phi^{-1}(n)(f - S_n(f))\right\|_{\Phi}'$$

$$= \inf\left\{k: \int_{-\pi}^{\pi}\Phi\left(\frac{n^r\Phi^{-1}(n)}{2k}(f - S_n(f))\right)dx < 1\right\}$$

$$< \inf\left\{k: C_{\Phi}\Phi\left(\frac{1}{k}\right)\int_{-\pi}^{\pi}\Phi\left(\frac{n^r\Phi^{-1}(n)}{2}(f - S_n(f))\right)dx < 1\right\}.$$

Let

$$R = R(\Phi, n, f) = C_{\Phi}\int_{-\pi}^{\pi}\Phi\left(\frac{n^r\Phi^{-1}(n)(f - S_n(f))}{2}\right)dx$$

then

$$\left\| \frac{n^r \Phi^{-1}(n)}{2} (f - S_n(f)) \right\|_\Phi' < \frac{1}{\Phi^{-1}(1/R)}.$$

Now we need only determine R.

Let χ_j denote the characteristic function of the set $(x_j - \delta, x_j + \delta)$, $j = 2, \ldots, m - 1$, of the set $(x_1 - \delta, x_1 + \delta) \cup (x_1 - \delta + 2\pi, \pi)$ for $j = 1$, and of the set $(x_m - \delta, x_m + \delta) \cup (-\pi, x_m + \delta - 2\pi)$ for $j = m$. Then

$$(f - S_n(f))(x) = \sum_{j=1}^m \chi_j(x) \sigma_j \varphi_{(r),n}(x - x_j) + \sum_{j=1}^m (1 - \chi_j(x)) \sigma_j \varphi_{(r),n}(x - x_j)$$

and

$$\frac{R}{C_\Phi} = \int_{-\pi}^\pi \Phi \left(\frac{n^r \Phi^{-1}(n)}{2} \sum_{j=1}^\infty \chi_j(x) \sigma_j \varphi_{(r),n}(x - x_j) \right.$$

$$\left. + \frac{n^r \Phi^{-1}(n)}{2} \sum_{j=1}^m (1 - \chi_j(x)) \sigma_j \varphi_{(r),n}(x - x_j) \right) dx$$

$$< \frac{1}{2} \int_{-\pi}^\pi \Phi \left(n^r \Phi^{-1}(n) \sum_{j=1}^m \chi_j(x) \sigma_j \varphi_{(r),n}(x - x_j) \right) dx$$

$$+ \frac{1}{2} \int_{-\pi}^\pi \Phi \left(n^r \Phi^{-1}(n) \sum_{j=1}^m (1 - \chi_j(x)) \sigma_j \varphi_{(r),n}(x - x_j) \right) dx$$

$$= \frac{1}{2} \sum_{j=1}^m \int_{x_j - \delta}^{x_j + \delta} \Phi \left(n^r \Phi^{-1}(n) \sigma_j \varphi_{(r),n}(x - x_j) \right) dx$$

$$+ O_{m,\delta,\sigma} \left(\Phi(\Phi^{-1}(n)/n) \right),$$

since, by Lemma 2.3,

$$\int_{-\pi}^\pi \Phi \left(n^r \Phi^{-1}(n)(1 - \chi_j(x)) \sigma_j \varphi_{(r),n}(x - x_j) \right) dx = O_{m,\delta,\sigma} \left(\Phi(\Phi^{-1}(n)/n) \right).$$

By Lemma 2.2,

$$\frac{1}{2} \int_{x_j - \delta}^{x_j + \delta} \Phi \left(n^r \Phi^{-1}(n) \sigma_j \varphi_{(r),n}(x - x_j) \right) dx$$

$$< \frac{C_\Phi}{2} \Phi(\sigma_j) \int_{x_j - \delta}^{x_j + \delta} \Phi \left(n^r \Phi^{-1}(n) \varphi_{(r),n}(x - x_j) \right) dx$$

$$< (C_\Phi/2) \Phi(\sigma_j) \left(S + O \left(\Phi(\Phi^{-1}(n)/n) \right) \right),$$

and we have, finally,

$$\frac{R}{C_\Phi} < \frac{C_\Phi}{2} \sum_{j=1}^m \Phi(\sigma_j) \left[S + O\left(\Phi\left(\frac{\Phi^{-1}(n)}{n} \right) \right) \right] + O_{m,\delta,\sigma}\left(\Phi\left(\frac{\Phi^{-1}(n)}{n} \right) \right)$$

$$= \frac{C_\Phi S}{2} \sum_{j=1}^m \Phi(\sigma_j) + O_{m,\delta,\sigma}\left(\Phi\left(\frac{\Phi^{-1}(n)}{n} \right) \right).$$

Thus,

$$\|f - S_n(f)\|_\Phi < \frac{2}{n^r \Phi^{-1}(n)} \cdot 1 \bigg/ \Phi^{-1}\left[\frac{1}{\frac{C_\Phi^2 S}{2} \sum_{j=1}^m \Phi(\sigma_j) + O_{m,\delta,\sigma}\left(\Phi\left(\frac{\Phi^{-1}(n)}{n} \right) \right)} \right]$$

as desired.

2.5 LEMMA. *Let Φ be an N-function and $f \in \mathcal{V}_\Phi^*$. Then there exists $a \neq 0$ such that*

$$\sup_{|h| < \delta} \int_0^{2\pi} \Phi(af(x+h) - af(x)) \, dx \leqslant 3\delta V_\Phi^{(2\delta)}(af).$$

PROOF. Since $f \in \mathcal{V}_\Phi^*$, there exist $k \neq 0$ such that $kf \in \mathcal{V}_\Phi$. If 1 is such a k, then $f \in \mathcal{V}_\Phi$ and let $a = 1$, otherwise let a be any k as above. Fix $\delta > 0$ and take any $h \in (0, \delta]$. Let n be the least integer greater than or equal to $2\pi/h$, and let $t_i = ih$, $i = 0, 1, \ldots, n - 1$, $t_n = 2\pi$, $t_{n+1} = 2\pi + h$. Further define

$$t_i' = \begin{cases} t_i, & i = 0, 1, \ldots, n - 1, n + 1, \\ t_{n-1} + h, & i = n. \end{cases}$$

$$\int_0^{2\pi} \Phi(af(t+h) - af(t)) \, dt = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \Phi(af(x+h) - af(t)) \, dt$$

$$< \sum_{i=1}^n (t_i - t_{i-1}) \sup_{t \in [t_{i-1}, t_i]} \Phi(af(t+h) - a(f(t)))$$

$$< h \sum_{i=1}^n \sup_{t \in [t_{i-1}, t_i]} \Phi(af(t+h) - af(t))$$

Dividing this sum into two parts, we let Σ_0 denote the sum over even i, and let Σ_1 denote the sum over odd i. Then

$$\int_0^{2\pi} \Phi(af(t+h) - af(t))\, dt \leqslant h\Sigma_0 \sup_{t \in [t_{i-1}, t_i]} \Phi(af(t+h) - af(t))$$

$$+ h\Sigma_1 \sup_{t \in [t_{i-1}, t_i]} \Phi(af(t+h) - af(t))$$

$$\leqslant h\Sigma_0 \sup_{x,y \in [t'_{i-1}, t'_{i+1}]} \Phi(af(x) - af(y))$$

$$+ h\Sigma_1 \lim_{x,y \in [t'_{i-1}, t'_{i+1}]} \Phi(af(x) - af(y)).$$

If 2π is an integral multiple of h then there is no overlapping of the intervals in each summation. Otherwise, if n is even, $[2\pi - t'_{n-2}, t'_n - 2\pi] \cap [0, t'_2] = [0, t'_1]$, so we let Σ'_1 be the sum over all odd i, except $n-1$; if n is odd then $[2\pi - t'_{n-1}, t'_{n+1} - 2\pi] \cap [0, t'_2] = [0, t'_1]$, so let Σ'_1 be the sum over all odd i except n.

Fix $\varepsilon > 0$. We may choose $x'_i, y'_i \in [t'_{i-1}, t'_{i+1}]$ for all i so that

$$\Sigma_0 \sup_{x,y \in [t'_{i-1}, t'_{i+1}]} \Phi(af(x) - af(y)) \leqslant \Sigma_0 \left[\Phi(af(x'_i) - af(y'_i)) + \varepsilon/n \right]$$

$$\leqslant \Sigma_0 \Phi(af(x'_i) - af(y'_i)) + \varepsilon,$$

$$\Sigma'_1 \sup_{x,y \in [t'_{i-1}, t'_{i+1}]} \Phi(af(x) - af(y)) \leqslant \Sigma'_1 \left[\Phi(af(x'_1) - af(y'_i)) + \varepsilon/n \right]$$

$$\leqslant \Sigma'_1 \Phi(af(x'_i) - af(y'_i)) + \varepsilon,$$

and, n even,

$$\sup_{x,y \in [t'_{n-2}, t'_n]} \Phi(af(x) - af(y)) \leqslant \Phi(af(x'_{n-1}) - af(y'_{n-1})) + \varepsilon/n$$

or, n odd,

$$\sup_{x,y \in [t'_{n-1}, t'_{n+1}]} \Phi(af(x) - af(y)) \leqslant \Phi(af(x'_n) - af(y'_n)) + \varepsilon/n.$$

Now, $|x'_i - y'_i| \leqslant 2h$, $i = 1, \ldots, n$, and so

$$\int_0^{2\pi} \Phi(af(x+h) - af(x))\, dx \leqslant h\big(\Sigma_0 \Phi(af(x_i) - af(y_i)) + \varepsilon\big)$$

$$+ h\big(\Sigma'_1 \Phi(af(x'_i) - af(y'_i)) + \varepsilon\big) + h\big(V_\Phi^{(2h)}(af) + \varepsilon/n\big)$$

$$\leqslant 3h\big(V_\Phi^{(2h)}(af) + \varepsilon\big).$$

Since ε is arbitrary, we have

$$\sup_{|h| < \delta} \int_0^{2\pi} \Phi(af(x+h) - af(x))\, dx \leqslant 3\delta V_\Phi^{(2\delta)}(af).$$

2.6 LEMMA. *Suppose Φ and Ψ are N-functions, Ψ is Δ_2, and* $\lim_{x \to 0} \Psi(x)/\Phi(x) = 0$. *Then* $\mathcal{V}_\Phi \subset W_\Psi^* = \{ f \in \mathcal{V}_\Psi : V_\Psi^*(f) = \Sigma_{j=1}^\infty \Psi(\sigma_j)$

where $\{\sigma_j\}$ is the sequence of jumps of $f\}$.

PROOF. Let $\varepsilon > 0$. Suppose $f \in \mathcal{V}_\Phi$, $\{x_k\}$ its points of discontinuity and $\{\sigma_k\}$ the corresponding sequence of jumps of f.

Define

$$g_m(x) = \sum_{k=1}^{m} \sigma_k \varphi_{(0),0}(x - x_k)$$

and let

$$h_m(x) = f(x) - g_m(x).$$

Since $\lim_{x \to 0} \Psi(x)/\Phi(x) = 0$, there exists $\lambda = \lambda(\varepsilon)$ such that, when $|x| < \lambda$,

(1) $$\Psi(x) < \varepsilon \Phi(x)/V_\Phi(f) + 2.$$

Now $f \in \mathcal{V}_\Phi$ implies that $\Sigma \Phi(\sigma_k) < \infty$ which implies $|\sigma_k| \to 0$. Thus, there is an $N = N(\lambda, \varepsilon)$ such that $\Sigma_{N+1}^{\infty} \Psi(\sigma_k) < \Psi(\lambda/4)$. We have that h_N is continuous at $x_1, \ldots, x_N$ and that the oscillation of h_N at $x \in [0, 2\pi]$ is less than $\lambda/4$, so there exists an $\eta_1 > 0$ so that $\omega(\eta_1, h_N) < \lambda/2$.

Next, since g_N is a piecewise linear function, there exists η_2 so that $|g_N(\beta) - g_N(\alpha)| < \lambda/2$ for $(\beta - \alpha) < \eta_2$ and α, β on the same interval of linearity. Let $\delta_0 = \min(\eta_1, \eta_2)$.

(2) $$|f(\beta) - f(\alpha)| \leqslant |g_N(\beta) - g_N(\alpha)| + |h_N(\beta) - h_N(\alpha)| < \lambda$$

whenever $|\beta - \alpha| < \delta_0$ and $\{x_i\}_{i=1}^{N} \cap [\alpha, \beta] = \varnothing$.

Let γ be the minimum length of the intervals of linearity of g_N, and select $\delta_1 < \min(\delta_0, \gamma)$ so that when

(3) $\alpha < x_i < \beta$ and $|\beta - \alpha| < \delta_1$,

$$\Psi(f(\beta) - f(\alpha)) < \Psi(\sigma_i) + \varepsilon/\big(N(V_\Phi(f) + 2)\big), \qquad i = 1, \ldots, N,$$

and when

(4) $x_i = \alpha$ or $x_i = \beta$, $|\beta - \alpha| < \delta_1$,

$$\Psi(f(\beta) - f(\alpha)) < \Psi(\sigma_i/2) + \varepsilon/\big(N(V_\Phi(f) + 2)\big).$$

We now choose a partition $\{y_j\}$ of $[0, 2\pi]$ with mesh less than δ_1, so that $\{x_i\}_{i=1}^{N} \cap \{y_j\} = \varnothing$. Denote by Σ_a the summation over all j where $\{x_i\}_{i=1}^{N} \cap [y_{j-1}, y_j] = \varnothing$, and by Σ_b the summation over all j where $\{x_i\}_{i=1}^{N} \cap [y_{j-1}, y_j] \neq \varnothing$. Then from (1) and (2),

$$\Sigma_a \Psi(f(y_j) - f(y_{j-1})) = \Sigma_a \frac{\Psi(f(y_j) - f(y_{j-1}))}{\Phi(f(y_j) - f(y_{j-1}))} \Phi(f(y_j) - f(y_{j-1}))$$

$$\leqslant \frac{\varepsilon}{V_\Phi(f) + 2} \Sigma_a \Phi(f(y_j) - f(y_{j-1})).$$

For Σ_b, we have, from (3), that,

$$\Sigma_b \Psi(f(y_i) - f(y_{j-1})) \leqslant \sum_{i=1}^{N} \left(\Psi(\sigma_i) + \varepsilon / \left(N(V_\Phi(f) + 2) \right) \right).$$

Hence,

$$\Sigma \Psi(f(y_j) - f(y_{j-1})) \leqslant \frac{\varepsilon}{V_\Phi(f) + 2} \Sigma_a \Phi(f(y_j) - f(y_{j-1}))$$

$$+ \frac{\varepsilon}{V_\Phi(f) + 2} + \sum_{j=1}^{N} \Psi(\sigma_i)$$

$$\leqslant \varepsilon + \sum_{i=1}^{\infty} \Psi(\sigma_i).$$

Now, on the other hand, suppose we choose a partition $\{z_j\}$ with mesh less than δ_1 such that $\{x_i\}_{i=1}^{N} \cap \{z_j\} \neq \varnothing$. Let

Σ_a denote the sum over all j where $\{x_i\}_{i=1}^{N} \cap [z_{j-1}, z_j] = \varnothing$,

Σ_b denote the sum over all j where $\{z_{j-1}, z_j\} \cap \{x_i\}_{i=1}^{N} \neq \varnothing$ and

Σ_c denote the sum over all j where $(z_{j-1}, z_j) \cap \{x_i\}_{i=1}^{N} \neq \varnothing$.

Proceeding as before, $\Sigma_a \Psi(f(z_j) - f(z_{j-1})) < \varepsilon V_\Phi(f)/(V_\Phi(f) + 2)$. If we denote by Σ_b' the sum over all i such that $x_i = z_j$ for some i and j, then

$$\Sigma_b \Psi(f(z_j) - f(z_{j-1})) \leqslant 2\Sigma_b' \left[\Psi(\sigma_i/2) + \varepsilon / \left(N(V_\Phi(f) + 2) \right) \right]$$

$$\leqslant \Sigma_b' \Psi(\sigma_i) + 2\varepsilon(\Sigma_b'(1)) / \left(N(V_\Phi(f) + 2) \right).$$

Letting Σ_c' denote the sum over all i such that $x_i \in (z_{j-1}, z_j)$ for some j, we have

$$\Sigma_c \Psi(f(z_j) - f(z_{j-1})) \leqslant \Sigma_c' \left[\Psi(\sigma_i) + \varepsilon / \left(N(V_\Phi(f) + 2) \right) \right].$$

Thus

$$\Sigma \Psi(f(z_i) - f(z_{i-1})) = \Sigma_a + \Sigma_b + \Sigma_c$$

$$\leqslant \frac{\varepsilon V_\Phi(f)}{V_\Phi(f) + 2} + \Sigma_b' \frac{2\varepsilon}{N(V_\Phi(f) + 2)} + \Sigma_c' \frac{\varepsilon}{N(V_\Phi(f) + 2)} + \sum_{i=1}^{N} \Psi(\sigma_i)$$

$$\leqslant \sum_{i=1}^{\infty} \Psi(\sigma_i) + \varepsilon.$$

Thus, for any partition $\{w_j\}$ of mesh less than δ_1,

$$\Sigma \Psi(f(w_j) - f(w_{j-1})) \leqslant \sum_{j=1}^{\infty} \Psi(\sigma_i) + \varepsilon$$

and

$$V_\Psi^*(f) \leqslant \sum_{i=1}^{\infty} (\sigma_i) + \varepsilon.$$

Since ε is arbitrary, and we also know that $V_\Psi^*(f) \geqslant \sum_{i=1}^\infty \Psi(\sigma_i)$ for all functions $f \in \mathcal{V}_\Psi$, where $\{\sigma_i\}$ is the sequence of jumps of f, the desired result follows.

2.7 LEMMA. *Suppose $f^{(r)} \in L_\Phi^*$, $r > 0$, then L_Φ^* reflexive implies*

$$\|f - S_n(f)\|_\Phi \leqslant \frac{c}{n^r} \omega_\Phi\left(\frac{1}{n}; f^{(r)}\right).$$

PROOF. For $f \in L_\Phi^*$, Ryan [10] has shown $\|S_n(f)\|_\Phi \leqslant c\|f\|_\Phi$, where c is independent of n and f if and only if L_Φ^* is reflexive. Suppose $T_n(x)$ is an nth degree trigonometric polynomial of best approximation to f and define

$$E_n(f)_\Phi = \|f(x) - T_n(x)\|_\Phi.$$

Reflexivity of L_Φ^* yields

(1) $$\|f - S_n(f)\|_\Phi \leqslant \|f - T_n\|_\Phi + \|S_n(T_n - f)\|_\Phi \leqslant (c + 1)E_n(f)_\Phi.$$

We now use a result which will be proved below:
For $f^{(r)} \in L_\Phi^*$, there exists $c = c(r, \Phi)$, such that

(*) $$E_n(f)_\Phi \leqslant \frac{c}{n^r} \omega_\Phi\left(\frac{1}{n}; f^{(r)}\right).$$

Equations (1) and (*) combined yield the result.

To prove (*) we need the following definitions and results from Sz.-Nagy [7]

1. Suppose $\Sigma_{jm} = \{\sigma_j, \ldots, \sigma_{m-1}, \tau_j, \ldots, \tau_{m-1}\}$ and $g \in L^1$, $g(x) \sim \sum_{k=j}^\infty \alpha_k \cos kx + \beta_k \sin kx$. Define a "regular approximation to $g(x)$" by

$$g_{m-1}(\Sigma_{jm}; x) = \sum_{k=j}^{m-1} \sigma_k(\alpha_k \cos kx + \beta_k \sin kx) + \tau_k(\beta_k \cos kx - \alpha_k \sin kx).$$

2. Henceforth, $\lambda = \{\lambda_k\}_{k=1}^\infty$ denotes a three-fold monotone sequence converging to zero ($\lambda_j > 0$, $\lambda_j \searrow 0$), for which $\sum(\lambda_k/k) < \infty$, and let δ be an arbitrary real number. If

$$f(x) \sim \sum_{k=j}^\infty a_k \cos kx + b_k \sin kx,$$

define

(1) $$(T_{\lambda\delta}^j f)(x) = \sum_{k=j}^\infty \lambda_k\big(a_k \cos(kx + \tfrac{1}{2}\delta\pi) + b_k \sin(kx + \tfrac{1}{2}\delta\pi)\big)$$

$$= \int H_{\lambda\delta}^j(y) f(x + y)\, dy$$

where

(2) $$H_{\lambda\delta}^j(y) = \sum_{k=j}^\infty \lambda_k \cos(kx - \tfrac{1}{2}\delta\pi).$$

The series on the right in (2) is a Fourier series converging to an integrable function, $H_{\lambda\delta}^j$, and the series in (1) is a convergent sequence almost everywhere.

3. THEOREM. *If* $\delta = 0$, *there exists a set* $\check{\Sigma}_{1m} = \{\check{\sigma}_1, \ldots, \check{\sigma}_{m-1}, \check{\tau}_1, \ldots, \check{\tau}_{m-1}\}$ *and a polynomial*

$$P_{\lambda 0}^{m-1}(x) = \sum_{k=1}^{m-1} \check{\xi}_k \cos kx + \check{\eta}_k \sin kx,$$

where for $1 \leqslant k \leqslant m-1$, $\check{\xi}_k = \lambda_k(1 - \check{\sigma}_k)$, $\check{\eta}_k = -\lambda_k \check{\tau}_k$, *such that*

$$\int_0^{2\pi} \left| H_{\lambda 0}^m(y) + P_{\lambda 0}^{m-1}(y) \right| dy = \frac{4}{\pi} \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\lambda_{(2\nu+1)m}}{2\nu+1}.$$

4. THEOREM. *If* $\delta = 1$, *there exists a set* $\hat{\Sigma}_{1m} = \{\hat{\sigma}_1, \ldots, \hat{\sigma}_{m-1}, \hat{\tau}_1, \ldots, \hat{\tau}_{m-1}\}$ *and a polynomial*

$$P_{\lambda 1}^{m-1}(x) = \sum_{k=1}^{m-1} \hat{\xi}_k \cos kx + \hat{\eta}_k \sin kx,$$

where for $1 \leqslant k \leqslant m-1$, $\hat{\xi}_k = \lambda_k \hat{\tau}_k$, $\hat{\eta}_k = \lambda_k(1 - \hat{\sigma}_k)$, *such that*

$$\int_0^{2\pi} \left| H_{\lambda 1}^m(y) + P_{\lambda 1}^{m-1}(y) \right| dy = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{\lambda_{(2\nu+1)m}}{2\nu+1}.$$

5. THEOREM. *For* δ *arbitrary there exists a polynomial* $P_{\lambda\delta}^{m-1}(x)$ *such that*

$$\int_0^{2\pi} \left| H_{\lambda\delta}^m(y) + P_{\lambda\delta}^{m-1}(y) \right| dy < \frac{4}{\pi} \left| \cos \tfrac{1}{2} \delta\pi \right| \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\lambda_{(2\nu+1)m}}{2\nu+1}$$

$$+ \frac{4}{\pi} \left| \sin \tfrac{1}{2} \delta\pi \right| \sum_{\nu=0}^{\infty} \frac{\lambda_{(2\nu+1)m}}{2\nu+1}.$$

Note that $H_{\lambda\delta}^m(x) = \cos \tfrac{1}{2} \delta\pi H_{\lambda 0}^m(x) + \sin \tfrac{1}{2} \delta\pi H_{\lambda 1}^m(x)$, so choose $P_{\lambda\delta}^{m-1}(x) = \cos \tfrac{1}{2} \delta\pi P_{\lambda 0}^{m-1}(x) + \sin \tfrac{1}{2} \delta\pi P_{\lambda 1}^{m-1}(x)$. Select

$$\Sigma_{1m}^* = \{\sigma_1^*, \ldots, \sigma_{m-1}^*, \tau_1^*, \ldots, \tau_{m-1}^*\}$$

so that if

$$P_{\lambda\delta}^{m-1}(x) = \sum_{k=1}^{m-1} \xi_k^* \cos kx + \eta_k^* \sin kx,$$

then

$$\xi_k^* = \lambda_k \cos \frac{\delta\pi}{2} - \lambda_k \left(\sigma_k^* \cos \frac{1}{2} \delta\pi - \tau_k^* \sin \frac{1}{2} \delta\pi \right),$$

$$\eta_k^* = \lambda_k \sin \frac{1}{2} \delta\pi - \lambda_k \left(\sigma_k^* \sin \frac{1}{2} \delta\pi + \tau_k^* \cos \frac{1}{2} \delta\pi \right).$$

6. If $f(x) = \int H_{\lambda\delta}^1(y) g(x + y) \, dy$, then

$$\left| f(x) - f_{m-1} (\Sigma_{1m}^*; x) \right| = \left| \int \big(H_{\lambda\delta}^m (y) + P_{\lambda\delta}^{m-1} (y) \big) g(x + y) \, dy \right|.$$

7. If $\lambda = \{ k^{-\alpha} \}_{k=1}^{\infty}$, for $\alpha > 0$, then λ is a three-fold monotone decreasing sequence, $k^{-\alpha} \searrow 0$, and $\Sigma k^{-\alpha-1} < \infty$. Let $\delta = -\alpha$. Then by 1–6, there exists Σ_{1m}^* so that

$$\int_0^{2\pi} \left| H_{\lambda,(-\alpha)}^m (y) + P_{\lambda,(-\alpha)}^{m-1} (y) \right| dy$$

$$< \frac{4}{\pi} \left| \cos \tfrac{1}{2} \alpha\pi \right| \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{(2\nu + 1)((2\nu + 1)m)^{\alpha}}$$

$$+ \frac{4}{\pi} \left| \sin \tfrac{1}{2} \alpha\pi \right| \sum_{\nu=0}^{\infty} \frac{1}{(2\nu + 1)((2\nu + 1)m)^{\alpha}}$$

$$< \frac{1}{m^{\alpha}} \left[\frac{4}{\pi} \left| \cos \tfrac{1}{2} \alpha\pi \right| \sum_{\nu=0}^{\infty} (-1)^{\nu} (2\nu + 1)^{-1-\alpha} \right.$$

$$\left. + \frac{4}{\pi} \left| \sin \tfrac{1}{2} \alpha\pi \right| \sum_{\nu=0}^{\infty} (2\nu + 1)^{-1-\alpha} \right]$$

$$< \frac{1}{m^{\alpha}} C_{\alpha}.$$

We continue with the proof of (∗). If f, g are as in 6, and $g \in L_{\Phi}$, then

$$\left\| f(x) - f_{m-1} (\Sigma_{1m}^*; x) \right\|_{\Phi}$$

$$= \sup \left\{ \int_0^{2\pi} v(x) \int_0^{2\pi} g(x + y) \big(H_{\lambda\delta}^m (y) + P_{\lambda\delta}^{m-1} (y) \big) \, dy \, dx : \right.$$

$$\left. \int \Phi^*(v(x)) \, dx < 1 \right\}$$

$$\leqslant \| g \|_{\Phi} \int_0^{2\pi} \left| H_{\lambda\delta}^m (y) + P_{\lambda\delta}^{m-1} (y) \right| dy.$$

Since $T_{\lambda,(-\alpha)}^1 g$, $\lambda = \{ k^{-\alpha} \}$, is just the fractional integral of g of order α, then $g = f^{(\alpha)}$, and, by 7

$$\left\| f(x) - f_{m-1} (\Sigma_1^*; x) \right\|_{\Phi} \leqslant \| f^{(\alpha)} \|_{\Phi} C_{\alpha} m^{-\alpha}.$$

Thus

$$E_m (f)_{\Phi} \leqslant C_{\alpha} m^{-\alpha} \| f^{(\alpha)} \|_{\Phi}.$$

Next, define

$$f(h; x) = \frac{1}{h} \int_{-h/2}^{h/2} f(x + t) \, dt,$$

and for $g \in L_\Phi$,

$$\left\| (g(h; x))' \right\|_\Phi = \frac{1}{h} \sup \left\{ \int v(x)(g(x + h/2) - g(x - h/2)) \, dx : \right.$$

$$\left. \int \Phi^*(v(x)) \, dx \leqslant 1 \right\}$$

$$= \frac{1}{h} \left\| g(x + h/2) - g(x - h/2) \right\|_\Phi \leqslant \frac{1}{h} \omega_\Phi(h; g),$$

and

$$\left\| g(x) - g(h; x) \right\|_\Phi = \sup \left\{ \int_0^{2\pi} v(x)(g(x) - g(h; x)) \, dx : \right.$$

$$\left. \int \Phi^*(v(x)) \, dx \leqslant 1 \right\}$$

$$= \sup \left\{ \int_0^{2\pi} v(x) \frac{1}{h} \int_{-h/2}^{h/2} (g(x) - g(x + t)) \, dt \, dx : \int \Phi^*(v(x)) \, dx \leqslant 1 \right\}$$

$$\leqslant h^{-1} \int_{-h/2}^{h/2} \left\| g(x) - g(x + t) \right\|_\Phi \, dt$$

$$\leqslant \omega_\Phi(h/2; g).$$

Noting that $(f(x) - f(h; x))^{(\alpha)} = f^{(\alpha)} - (f(h; x))^{(\alpha)} = f^{(\alpha)} - f^{(\alpha)}(h; x)$ almost everywhere, we have

$$E_n(f)_\Phi \leqslant E_n(f - f(h; \cdot))_\Phi + E_n(f(h; \cdot))_\Phi$$

$$\leqslant \frac{c_\alpha}{n^\alpha} \left\| f^{(\alpha)} - (f(h; \cdot))^{(\alpha)} \right\|_\Phi + \frac{c_{\alpha+1}}{n^{\alpha+1}} \left\| (f(h; \cdot))^{(\alpha+1)} \right\|_\Phi$$

$$= \frac{c_\alpha}{n^\alpha} \left\| f^{(\alpha)} - f^{(\alpha)}(h; \cdot) \right\|_\Phi + \frac{c_{\alpha+1}}{n^{\alpha+1}} \left\| (f^{(\alpha)}(h; \cdot)) \right\|_\Phi$$

$$\leqslant \frac{c_\alpha}{n^\alpha} \omega_\Phi(h/2; f^{(\alpha)}) + \frac{c_{\alpha+1}}{n^{\alpha+1}} \cdot h^{-1} \omega_\Phi(h; f^{(\alpha)}).$$

Now, letting $h = 1/h$, we have

$$E_n(f)_\Phi \leqslant (c_\alpha + c_{\alpha+1}) n^{-\alpha} \omega_\Phi(n^{-1}; f^{(\alpha)}),$$

which is (∗) when $\alpha = r$.

2.8 LEMMA. *Suppose* $f \in \mathcal{V}_\Phi^{(r)}$, $r > 0$, Φ *an N-function. If* Ψ *is a* Δ' *N-function and* L_Φ^* *is reflexive then*

$$\|f - S_n(f)\|_\Psi'$$

$$\leqslant \frac{c}{n^r \Psi^{-1}(n)} \left[1 \Big/ \left(\Psi^{-1}\left(\frac{1}{cV_\Phi^{(2/n)}(f^{(r)})} \right) \Psi^{-1}\left(\frac{1}{\gamma(\Psi, \Phi, f^{(r)}, n^{-1})} \right) \right) \right]$$

where

$$\gamma(\Psi, \Phi, g, h) = \sup\left\{ \frac{\Psi(g(x+t) - g(x))}{\Phi(g(x+t) - g(x))} : x \in [0, 2\pi], |t| \leqslant h \right\}.$$

PROOF. Suppose $g(x) \in \mathcal{V}_\Phi$. Then using Lemma 2.5,

$$\|g(x+h) - g(x)\|_\Psi' = \inf\left\{ k: \int \Psi\left(\frac{g(x+h) - g(x)}{k} \right) dx \leqslant 1 \right\}$$

$$\leqslant \inf\left\{ k: c\Psi(1/k) \int \frac{\Psi(g(x+h) - g(x))}{\Phi(g(x+h) - g(x))} \cdot \Phi(g(x+h) - g(x)) \, dx \leqslant 1 \right\}$$

$$\leqslant \inf\left\{ k: c\Psi(1/k)\gamma(\Psi, \Phi, g, h) \int \Phi(g(x+h) - g(x)) \, dx \leqslant 1 \right\}$$

$$\leqslant \inf\{ k: c\Psi(1/k)\gamma(\Psi, \Phi, g, h)3hV_\Phi^{(2h)}(g) \leqslant 1 \}.$$

Thus

$$\|g(x+h) - g(x)\|_\Psi'$$

$$\leqslant 1 \Big/ \left(\Psi^{-1}\left(\frac{1}{h} \right) \Psi^{-1}\left(\frac{1}{\gamma(\Psi, \Phi, g, h)} \right) \Psi^{-1}\left(\frac{c}{V_\Phi^{(2h)}}(g) \right) \right).$$

By Lemma 2.7

$$\|f - S_n(f)\|_\Psi \leqslant \frac{c_r^\Psi}{n^r} \omega_\Psi\left(\frac{1}{n} ; f^{(r)} \right)$$

$$\leqslant \frac{c_r}{n^r \Psi^{-1}(n)} \cdot \frac{1}{\Psi^{-1}\left(c/V_\Phi^{(2/n)}(f^{(r)}) \right)} \cdot \frac{1}{\Psi^{-1}\left(1/\gamma(\Psi, \Phi, f^{(r)}, 1/n) \right)}.$$

We now return to the proof of the main theorem of this section.

PROOF. Let

$$u_m(x) = f^{(r)}(x) - \sum_{j=1}^m \sigma_j \varphi_{(0),0}(x - x_j).$$

The oscillation of $u_m(x)$ at x goes to zero uniformly in x as $m \to \infty$ since $\Sigma\Phi(\sigma_j) < \infty$. We let γ be as in Lemma 2.8 and select $\varepsilon > 0$. Then there exists

an m_0 and $\delta_0 > 0$ such that for all $m \geq m_0$, $\delta \leq \delta_0$,

$$\gamma(\Psi, \Phi, u_m/2, \delta) < 1/\Psi(1/\varepsilon)$$

since $\omega(u_m/2, t) \to 0$ as $m \to \infty$ and $t \to 0$. Consider

$$g(x) = \sum_{j=1}^{m_0} \sigma_j \varphi_{(r),0}(x - x_j)$$

and let $h(x) = f(x) - g(x)$. Then $h^{(r)}(x) = u_{m_0}(x)$. Now

$$\|f - S_n(f)\|'_\Psi \leq \|g - S_n(g)\|'_\Psi + 2\|h/2 - S_n(h/2)\|'_\Psi.$$

Applying Lemma 2.8 and Lemma 2.4 with δ' chosen as in that lemma,

$$(*) \quad \|f - S_n(f)\|'_\Psi < \frac{2}{n^r \Psi^{-1}(n)} \cdot \left[1 \Big/ \Psi^{-1}\left[\frac{1}{\frac{C_\Psi^2 S}{2} \sum_{j=1}^{m_0} \Psi(\sigma_j) + O_{m_0, \delta', \sigma}\left(\Psi\left(\frac{\Psi^{-1}(n)}{n}\right)\right)} \right] \right]$$

$$+ \frac{c}{n^r \Psi^{-1}(n)} \frac{1}{\Psi^{-1}(c/V_\Phi^{(2/n)}(h^{(r)}/2))} \cdot \frac{1}{\Psi^{-1}(1/\gamma(\Psi, \Phi, h^{(r)}/2, 1/n))}.$$

Now, select $n > 1/\delta_0$, which implies

$$\gamma\left(\Psi, \Phi, \frac{u_{m_0}}{2}, \frac{1}{n}\right) < \frac{1}{\Psi(1/\varepsilon)}$$

and, therefore,

$$\varepsilon > 1/\Psi^{-1}(1/\gamma(\Psi, \Phi, h')/2, 1/n).$$

Thus,

$$\|f - S_n(f)\|'_\Psi < \frac{2}{n^r \Psi^{-1}(n)} \cdot \left[1 \Big/ \Psi^{-1}\left[\frac{1}{\frac{C_\Psi^2 S}{2} \sum_{j=1}^{\infty} \Psi(\sigma_j) + O_{m_0, \delta', \sigma}\left(\Psi\left(\frac{\Psi^{-1}(n)}{n}\right)\right)} \right] \right]$$

$$+ \frac{c\varepsilon}{n^r \Psi^{-1}(n)\Psi^{-1}(c/V_\Phi^{(2/n)}(h^{(r)}/2))}.$$

Now, suppose $\{\sigma_i\}_{i=1}^{\infty} \neq \varnothing$. Since $\Psi(\Psi^{-1}(n)/n) \to 0$ as $n \to \infty$, we have

$$\varliminf_{n \to \infty} \left[\Psi^{-1}\left[1 \Big/ \left[\frac{C_\Psi^2 S}{2} \sum_{j=1}^{\infty} \Psi(\sigma_j) + O_{m_0, \delta', \sigma}\left(\Psi\left(\frac{\Psi^{-1}(n)}{n}\right)\right) \right] \right] \right]^{-1}$$

$$= \left[\Psi^{-1}\left[1 \Big/ \left(\frac{C_\Psi^2 S}{2} \sum_{j=1}^{\infty} \Psi(\sigma_j) \right) \right] \right]^{-1}$$

and

$$\varlimsup_{n\to\infty} \frac{c\varepsilon}{\Psi^{-1}\left(c/V_{\Phi}^{(2/n)}(h^{(r)}/2)\right)} = \frac{c\varepsilon}{\Psi^{-1}\left(c/V_{\Phi}^{*}(h^{(r)}/2)\right)}.$$

Since we know that

$$V_{\Phi}^{*}(g^{(r)}) = \sum_{j=1}^{m_0} \Phi(\sigma_j) < V_{\Phi}^{*}(f^{(r)}),$$

and $(f^{(r)} - g^{(r)})/2 = h^{(r)}/2$, we have

$$V_{\Phi}^{*}(h^{(r)}/2) \leqslant \tfrac{1}{2} V_{\Phi}^{*}(f^{(r)}) + \tfrac{1}{2} V_{\Phi}^{*}(g^{(r)}) < V_{\Phi}^{*}(f^{(r)}).$$

Hence,

$$\varlimsup_{n\to\infty} n^r \Psi^{-1}(n) \| f - S_n(f) \|_{\Psi}'$$

$$< 1 \Bigg/ \Psi^{-1}\left[\frac{1}{\dfrac{C_{\Psi}^2 S}{2} \displaystyle\sum_{j=1}^{\infty} \Psi(\sigma_j)} \right] + \frac{c\varepsilon}{\Psi^{-1}(c/V_{\Phi}^{*}(f))}$$

where the right-hand side has no dependence on m_0. Since ε is arbitrary, we have

$$\varlimsup_{n\to\infty} n^r \Psi^{-1}(n) \| f - S_n(f) \|_{\Psi}' < 1 \Bigg/ \Psi^{-1}\left[\frac{1}{\dfrac{C_{\Psi}^2 S}{2} \displaystyle\sum_{j=1}^{\infty} \Psi(\sigma_j)} \right].$$

If $f^{(r)}$ is continuous, we note that $g \equiv 0$ and (*) is replaced by

$$n^r \Psi^{-1}(n) \| f - S_n(f) \|_{\Psi}'$$

$$< \frac{c}{\Psi^{-1}\left(c/V_{\Phi}^{(2/n)}(f^{(r)})\right)} \cdot 1 \Bigg/ \Psi^{-1}\left(\frac{1}{\gamma(\Psi, \Phi, f^{(r)}, 1/n)} \right)$$

$$= o(1) \quad \text{as } n \to \infty.$$

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